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Calculus of Variations

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Global curvature for surfaces and area minimization under a thickness constraint

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Abstract Motivated by previous work on elastic rods with self-contact, involving the concept of the global radius of curvature for curves (as defined by Gonzalez and Maddocks), we define the *global radius of curvature* $\triangle[X]$ for a wide class of continuous parametric surfaces X for which the tangent plane exists on a dense set of parameters. It turns out that in this class of surfaces a positive lower bound $\triangle[X] \geq \theta > 0$ provides, naively speaking, the surface with a thickness of magnitude θ; it serves as an excluded volume constraint for X, prevents self-intersections, and implies that the image of X is an embedded C^1-manifold with a Lipschitz continuous normal. We also obtain a convergence and a compactness result for such thick surfaces, and show one possible application to variational problems for embedded objects: the existence of ideal surfaces of fixed genus in each isotopy class.

The proofs are based on a mixture of elementary topological, geometric and analytic arguments, combined with a notion of the reach of a set, introduced by Federer in 1959.

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1 Introduction

1.1 Physical and geometric motivation

Physical surfaces such as sheets of paper, thin elastic plates, pieces of cloth, or aluminium foil often undergo large deformations in space so that different parts of the same object touch each other. These self-contact phenomena can also be observed on various smaller length scales, especially in biological systems, e.g., pinched skin tissue, buckled membranes, conformations of lipid vesicles under thermal influence [29]. The underlying common feature of all these examples is that of a surface with a small but positive thickness reflecting the fact that interpenetration of matter is impossible.

The mathematical modelling of the intuitively obvious mechanism of self-avoidance is a challenging task: one needs an analytically tractable notion of thickness for surfaces, which in particular should be accessible to variational methods in order to deal with energy minimization problems in the framework of nonlinear elasticity. Moreover, surfaces with positive thickness are embedded; hence a suitable notion of self-avoidance should also lead to a novel treatment of classical geometric boundary value problems such as the Plateau problem or free and semi-free problems in the class of *embeddings*. This would produce physically relevant solutions of fixed topological type without self-intersections – in contrast to the classical solutions, where one frequently encounters non-embedded solutions due to the geometry of the boundary configurations, see the discussion on minimal surfaces in [6, Ch. 4.10].

Our aim is to introduce and investigate a purely geometric notion of thickness for a large class of (nonsmooth) parametric surfaces suitable for the calculus of variations. Motivated by the second author's previous cooperations on elastic rods with self-contact [15, 26–28], which involved the concept of the global radius of curvature for curves as suggested by Gonzalez and Maddocks [14], we define the *global radius of curvature for surfaces*. Most results of the present paper have already been announced (without proofs) in [31].

1.2 Brief discussion of results

The main idea can be sketched as follows. Take a continuous parametric surface $X : \mathbb{R}^2 \supset \mathbb{B}^2 \to \mathbb{R}^3$ (with possibly infinite area) which possesses a tangent plane on a dense subset $G \subset \mathbb{B}^2$ which may even have zero measure. Consider the radii of all spheres touching the surface $X(\overline{\mathbb{B}}^2)$ in one of these points $X(w)$, $w \in G$, and containing at least one other point of the surface. We define the infimum of these radii as the global radius of curvature $\triangle[X]$ of the surface X. It turns out that a positive lower bound on $\triangle[X]$ serves as an *excluded volume constraint* for the surface X.

In fact, one of our main results is that $\triangle[X] \geq \theta > 0$ implies that $X(\overline{\mathbb{B}}^2)$ is a $C^{1,1}$-manifold with boundary, where the domain size and the $C^{1,1}$-norms of the local graph representations of $X(\overline{\mathbb{B}}^2)$ are uniform and depend solely on the constant θ (Theorems 5.1 and 5.2). This result requires careful analysis of the normal in the interior and near the boundary, since the set $\mathbb{B}^2 \setminus G$ of bad points without a tangent plane is allowed to have full measure. In view of applications in the calculus of variations we prove that the excluded volume constraint in terms of the global radius of curvature is stable under pointwise convergence of parametrizations, see Theorem 6.6. Moreover, assuming a uniform upper bound on area and a uniform positive lower bound on the global radius of curvature for a family of surfaces we can prove the existence of a C^1-convergent subsequence to a limit manifold of class $C^{1,1}$, again with uniform control on the local graph representations (Theorem 6.4). This compactness result may be a crucial step towards the study of variational problems for embedded surfaces in geometry and nonlinear elasticity. In fact, we show that Theorem 6.4 may be used to prove the existence of *ideal surfaces* of fixed genus in each isotopy class, see Sect. 7. (The term 'ideal' is used to describe the 'thickest' surface having fixed area and prescribed isotopy class.) Let us also mention that our results carry over to arbitrary co-dimension and are not restricted to disk-type surfaces.

The presentation is structured as follows: In Sect. 2 we give the precise definitions of the class of admissible surfaces, of the global radius of curvature for surfaces and provide simple analytical and topological consequences. In Sect. 3 we prove a priori estimates for the normal line depending only on a positive lower bound for the global radius of curvature. We extend these results up to the boundary in Sect. 4, before we study the structure of the image to prove that such a surface is a $C^{1,1}$-manifold in Sect. 5. Section 6 contains the convergence and compactness results, which are applied in our existence proof of ideal surfaces in Sect. 7.

Before passing to the details, let us discuss some earlier papers related to our work.

1.3 Other approaches to thickness of surfaces

An alternative method to prevent a surface from self-intersecting is to introduce explicit repulsive forces between pairs of points on the surface. Based on this idea Kusner and Sullivan [18] suggested a Möbius invariant *knot energy* for k-dimensional submanifolds in $\mathbb{R}^n$ without boundary. These highly singular potential energies, however, require some regularization to account for adjacent points on the surface and, apart from the one-dimensional case of knot energies for curves [25], [11], [17], there are no analytical results regarding existence of minimizers or their regularity. Banavar, Gonzalez, Maddocks and Maritan [2] proposed so-called *many-body potentials*, replacing the Euclidean distance between two points by geometric multipoint functions on curves, or *tangent-point distances* for surfaces as Lagrangians for multiple integrals, in order to avoid the technical difficulties arising from the singularities in the potential, and to introduce an intrinsic length-scale for thickness. Although not stated explicitly in [2] Banavar et al. clearly had the concept of global curvature for smooth surfaces based on tangent-point distances in mind from which their many-body potentials arise.

Apart from numerical investigations in the protein science [3] based on these potentials, however, there are, to the best of our knowledge, no analytical results in this direction, with one exception: For a particular example of a three-body potential, the so-called *total Menger curvature* on one-dimensional sets, there is a remarkable regularity result of Léger [19] motivated by removability problems for bounded analytical functions in the complex plane (namely, the Vitushkin conjecture and its solution by David). Léger proved that a Borel set E with bounded total Menger curvature and with positive and finite one-dimensional Hausdorff measure is actually contained in the union of Lipschitz graphs apart from a measure zero set. He also claimed an analogous result for higher dimensional objects without giving the proof.

Another contribution to thickness of surfaces in terms of the classical injectivity radius and the geometric focal distance for $C^{1,1}$-smooth submanifolds without boundary is given by the work of Durumeric [9]. He proves among other things a compactness result based on Gromov's compactness theorem, and he provides upper bounds on the diffeomorphism and isotopy types for $C^{1,1}$-submanifolds with a uniform lower bound on the injectivity radius.

There are other papers that investigate surfaces under various weak assumptions imposed on geometric quantities. Semmes [30] considered hypersurfaces M^d in $\mathbb{R}^{d+1}$ whose normals have small norm in the space BMO of functions of bounded mean oscillation (such surfaces can twist and spiral, and be far from being graphs). He proved that each such M is a chord-arc surface with small constant, i.e. for each $x \in M$ and each $R > 0$, the intersection $B_R(x) \cap M$ stays close to the hyperplane that passes through x and is perpendicular to the mean value of the normal, $n_{x,r} = \fint_{B_R(x)} n(y)dy$, taken w.r.t. the surface measure on M. Toro [32] proved that surfaces with generalized fundamental form in L^2 are Lipschitz manifolds (as a consequence, the graph of every function $u \in W^{2,2}(\Omega, \mathbb{R})$, where $\Omega \subset \mathbb{R}^2$, can be parametrized by a bi-Lipschitz map). Her work was later generalized by Müller and Šverák [23] who give a sharp condition on the L^2-norm of the second fundamental form that guarantees that a complete, connected, noncompact surface immersed in $\mathbb{R}^d$ is embedded.

For surfaces S homeomorphic to $\mathbb{R}^2$ these results were improved by Bonk and Lang [5], who, to answer a conjecture of Fu [12], considered a very rich class of Alexandrov surfaces, with a notion of integral curvature defined as a signed measure μ on S (if S is smooth, then for each $A \subset S$ the value $\mu(A)$ is equal to the integral of Gaussian curvature over A w.r.t. the surface measure). They proved that if $\mu^+(S) < 2\pi$ and $\mu^-(S)$ is finite, then S is bi-Lipschitz equivalent to the plane. The bound 2π in their result is sharp; to see this one considers a semi-infinite cylinder with a hemisphere attached to its end.

Our work is also related to Federer's notion of *sets of positive reach* introduced in his seminal paper [10] on curvature measures. In fact, Sect. 4 of that paper provides valuable tools for the proofs of our convergence and compactness results, see Theorems 6.6 and 6.4.

2 Admissible mappings and basic definitions

Throughout the paper we assume that $X : \overline{\mathbb{B}}^2 \to \mathbb{R}^3$ is continuous up to the boundary of the unit disk $\mathbb{B}^2 := \{w \in \mathbb{R}^2 : |w| < 1\}$. We also require X to be differentiable in the classical sense at all points $w \in G$, where G is a dense subset of $\mathbb{B}^2$ (which of course may depend on X and may even have measure zero), and we impose the condition

$$\operatorname{rank} DX(w) = 2 \qquad \text{for all } w \in G, \tag{2.1}$$

so that the affine tangent plane $T_w X := X(w) + DX(w)(\mathbb{R}^2)$ is a well-defined two-dimensional affine plane. In the sequel each $w \in G \subset \mathbb{B}^2$ is called a *good parameter*, and G is referred to as the set of good parameters.

Each such mapping will be called *admissible*. The class of all admissible mappings is denoted by $\mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$. It is clear that a mapping $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$ can *a priori* have infinite area. On the other hand, if $X \in C^1(\mathbb{B}^2, \mathbb{R}^3) \cap C^0(\overline{\mathbb{B}}^2, \mathbb{R}^3)$ is an immersion, then X is contained in $\mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$.

Finally, note that if Σ is an arbitrary two-dimensional Riemannian manifold with or without boundary then the class $\mathcal{A}(\Sigma, \mathbb{R}^3)$, and in fact also $\mathcal{A}(\Sigma, \mathbb{R}^d)$, where $d \geq 3$, can be defined in a similar way.

Remark. To give an example of a well investigated class of (nonsmooth) mappings where most of the above assumptions are automatically satisfied, we recall here the definition and a handful of properties of n-*absolutely continuous* functions denoted by AC^n. (In our setting $n = 2$.) Let $\Omega \subset \mathbb{R}^n$. One says, see Malý [21], that $f \in AC^n(\Omega, \mathbb{R}^d)$ whenever for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $k \in \mathbb{N}$ and every finite family of pairwise disjoint balls $B_1, \ldots, B_k$ in Ω the following is satisfied:

$$\sum_{i=1}^{k} \mathcal{L}^n(B_i) < \delta \quad \Rightarrow \quad \sum_{i=1}^{k} \Big(\operatorname*{osc}_{B_i} f\Big)^n < \varepsilon,$$

where

$$\operatorname*{osc}_{A} f = \sup_{x,y \in A} |f(x) - f(y)|$$

stands for the oscillation of f on A and $\mathcal{L}^n$ denotes the Lebesgue measure. Obviously, such mappings are continuous. Malý proves that n-absolute continuity (for mappings $f\colon \Omega \to \mathbb{R}^d$, where d can be arbitrary) implies weak differentiability with gradient in L^n (so that for $n = 2$ the area of f is finite!) and classical differentiability almost everywhere. Moreover, for $d \geq n$ the Lusin condition is satisfied, i.e. $\mathcal{H}^n(f(E)) = 0$ whenever $\mathcal{L}^n(E) = 0$, and the area formula holds.

Thus, for planar domains Ω, $AC^2(\Omega, \mathbb{R}^3)$ is a proper subset of the Sobolev space of $W^{1,2}(\Omega, \mathbb{R}^3)$. The latter space obviously contains discontinuous mappings; it also contains mappings which are continuous but nowhere differentiable in the classical sense. On the other hand, for every bounded domain Ω we have

$$\bigcup_{p>2} W^{1,p}(\Omega, \mathbb{R}^3) \subset AC^2(\Omega, \mathbb{R}^3), \tag{2.2}$$

so that the class AC^2 is indeed larger than any of the Sobolev spaces $W^{1,p}$, $p > 2$. In fact, the inclusion in (2.2) is proper: the mapping

$$\mathbb{B}^2 \ni w \longmapsto f(w) = \frac{w}{|w|} \frac{1}{\log(1 + |w|^{-1})} \in \mathbb{R}^2 \subset \mathbb{R}^3$$

is of class $AC^2 \setminus W^{1,p}$ for every $p > 2$ (we leave easy computational details to the reader).

Let X be an admissible mapping. For all $w \in \mathbb{B}^2$ and all *good* $w' \in \mathbb{B}^2$ (i.e., $w' \in G$) we set

$$n(w') := \frac{X_u(w') \wedge X_v(w')}{|X_u(w') \wedge X_v(w')|}.$$

Then we define for $w \in \mathbb{B}^2$, $w' \in G \subset \mathbb{B}^2$

$$r(X(w); X(w'), DX(w')) \\ := \begin{cases} 0 & \text{if } X(w) = X(w'), \\ \infty & \begin{array}{l}\text{if } X(w) \in T_{w'}X \\ \text{and } X(w) \neq X(w'),\end{array} \\ \dfrac{|X(w) - X(w')|}{2\left|n(w') \cdot \frac{X(w)-X(w')}{|X(w)-X(w')|}\right|} & \text{in the remaining cases.} \end{cases} \tag{2.3}$$

In plain words, $r(x, y, p)$ is the radius of the unique sphere through the points $x, y \in \mathbb{R}^3$ tangent to the affine plane $y + p(\mathbb{R}^2)$, where p is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$ with rank $p = 2$. This radius becomes infinite when the vector $x - y \neq 0$ lies in the plane $p(\mathbb{R}^2)$, and is set to be zero if $x = y$.

Definition 2.1 For arbitrary $w \in \mathbb{B}^2$ we call

$$\rho[X](w) := \inf_{w' \in G} r(X(w); X(w'), DX(w'))$$

the *global radius of curvature of X at w*, and

$$\triangle[X] := \inf_{w \in \mathbb{B}^2} \rho[X](w)$$

the *global radius of curvature of X*.

The intuitive idea behind this concept is that a positive lower bound $\theta > 0$ on $\triangle[X]$ will allow us to place a pair of open balls of radius at least θ at each point of the surface "touching" the surface from both sides without intersecting it. From this we can infer that any surface X with $\triangle[X] \geq \theta$ satisfies the *excluded volume constraint* as described in the introduction. In particular, $X(\mathbb{B}^2)$ is an embedded surface in $\mathbb{R}^3$. Of course, all this requires proof, especially since only good points $X(w)$, $w \in G$ – hence possibly only countably many surface points – can be used for this construction. A detailed investigation of the geometric and analytical properties of surfaces with positive global radius of curvature is carried out in Sects. 3–6.

As a first consequence of Definition 2.1 let us note at this point that for any $w \in \mathbb{B}^2$ with $\rho[X](w) > 0$ one has $X(w) \neq X(w')$ for all $w' \in G$. Consequently, if $\triangle[X] > 0$, then $X(w) = X(\tilde{w})$ implies either $w = \tilde{w}$ or that both w and $\tilde{w}$, are "bad" parameters, i.e., $w, \tilde{w} \in \mathbb{B}^2 \setminus G$.

Moreover, if $w' \in G$ and $\triangle[X] \geq \theta > 0$ then the two open balls B_1, B_2 of radius θ centered at $X(w') + \theta n(w')$ and $X(w') - \theta n(w')$ do not intersect the surface $X(\mathbb{B}^2)$, since otherwise we could find a point $X(w)$ such that

$r(X(w); X(w'), DX(w')) < \theta$ contradicting our assumption on $\triangle[X]$. We shall sometimes call B_1, B_2 *excluded* or *forbidden* balls.

Since G was only required to be a dense (so possibly just countable) subset of $\mathbb{B}^2$, the set $\mathbb{B}^2 \setminus G$ of bad parameters could have full measure, which necessitates a careful investigation of the geometric and analytical properties of surfaces with positive global curvature, see Sects. 3–6.

Remark. Notice that for admissible mappings $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^d)$, $d > 3$, the global radius of curvature $\triangle[X]$ can be defined analogously. The definition of $r(x, y, p)$ remains unchanged. There is, however, one notable difference. For every good parameter w' in the domain we have now — instead of two excluded touching balls centered at $X(w') \pm \theta n(w')$ — a forbidden *region*

$$U_{w'} = \bigcup_{q \in S_{\theta,w'}} B_\theta(q),$$

where the set of centers

$$S_{\theta,w'} := \mathbb{S}_\theta^{d-1}(X(w')) \cap N_{w'}X$$

is given by the intersection of the round $(d-1)$-dimensional sphere

$$\mathbb{S}_\theta^{d-1}(X(w')) = \{s \in \mathbb{R}^d \ : \ |s - X(w')| = \theta\}$$

with the affine normal space $N_{w'}X = X(w') + (DX(w')(\mathbb{R}^2))^\perp$. Thus, $U_{w'}$ looks, roughly speaking, like a thick degenerate doughnut. We have $\dim N_{w'}X = d-2$ for good w', therefore $S_{\theta,w'}$ is in fact a $(d-3)$-dimensional sphere in $N_{w'}X$. (Note that for $d = 3$ the centers of B_1, B_2 do form a zero dimensional sphere contained in the normal line.) Analogously to the co-dimension 1 case, $U_{w'}$ touches $T_{w'}X$ and is excluded for X, i.e. $X(\mathbb{B}^2) \cap U_{w'}$ is empty.

3 Interior continuity of the normal

Let $X : \mathbb{B}^2 \to \mathbb{R}^3$ be an admissible mapping, i.e., $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$, with the property that $\triangle[X] \geq \theta$, and let $\varrho \in (0, \theta)$. Assume that $w \in \mathbb{B}^2$ is a good parameter, i.e., $w \in G$. Let

$$\ell(w) :\ = \{X(w) + tn(w) \ : \ t \in \mathbb{R}\}$$

be the (affine) normal line to X at w. We set

$$\begin{aligned} d(w) \ &: \ = \operatorname{dist}(X(w), X(\partial\mathbb{B}^2)) \\ C_\varrho(w) \ &: \ = \{p \in \mathbb{R}^3 \,|\, \operatorname{dist}(p, \ell(w)) = \varrho\}, \end{aligned}$$

and write π_w to denote the orthogonal projection onto the (affine) tangent plane T_wX with center $X(w)$.

To show that the normal direction to X is uniformly continuous on compact subsets of $\mathbb{B}^2$, we need the following technical definition.

Definition. We say that X has the *ϱ-stretching property* at $w \in G \subset \mathbb{B}^2$ iff

$$\pi_w\big(C_\varrho(w) \cap X(\mathbb{B}^2) \cap B_{2\varrho}(X(w))\big)$$

is a circle of radius ϱ in the tangent plane $T_w X$ with center $X(w)$.

Lemma 3.1 *Assume that $\triangle[X] \ge \theta > 0$. If $w \in G$ and $\varrho \in (0, \varrho_0]$, where $\varrho_0 := \min\big(\theta, d(w)/2\big)$, then X has the ϱ-stretching property at w.*

This Lemma means that at every good point w the image of X stretches away from $X(w)$ in all directions parallel to $T_w X$, as long as the distance from $X(w)$ is comparable to θ. Intuitively, a surface with $\triangle[X] \ge \theta$ cannot fold abruptly at length scales much smaller than θ: close to *every* straight line through $X(w)$ in $T_w X$ intersected with $B_r(X(w))$ we see points of the surface, as long as $r < \theta \le \triangle[X]$ and the boundary $X(\partial\mathbb{B}^2)$ is far away.

Proof. Fix $w \in G$ and $\varrho \in (0, \varrho_0]$. Without loss of generality we assume that $X(w) = 0 \in \mathbb{R}^3$ and $n(w) = (0, 0, 1)$.

Let $B_1 = B_\theta(0, 0, \theta)$ and $B_2 = B_\theta(0, 0, -\theta)$; we have $X(\mathbb{B}^2) \cap (B_1 \cup B_2) = \emptyset$. Since $\operatorname{rank} DX(w) = 2$, the curve $X(\partial B_\delta(w))$ is, for some sufficiently small $\delta \in (0, \varrho)$, linked with the normal line $\ell(w)$.

Now, let $I \subset C_\varrho(w) \setminus (B_1 \cup B_2)$ be a fixed (but otherwise arbitrary) line segment contained in $B_{2\varrho}(0)$ and having its endpoints on ∂B_1 and ∂B_2. To show that $X(\mathbb{B}^2) \cap I$ is nonempty, consider a homotopy $(\gamma_s)_{s\in[0,1]}$ from $\gamma_0 = X(\partial B_\delta(w))$ to $\gamma_1 = X(\partial\mathbb{B}^2(0))$, defined as a composition of X with a homotopy from $\partial B_\delta(w)$ to $\partial\mathbb{B}^2$ in $\overline{\mathbb{B}}^2 \setminus B_\delta(w)$.

Let σ be the closed curve consisting of I and two segments that join the endpoints of I to $0 = X(w)$. The curves γ_0 and σ are linked, whereas γ_1 and σ are not linked, for otherwise we would have $\operatorname{dist}(X(w), \gamma_1) < \varrho$, contrary to the definition of ϱ_0, see Fig. 1.

It follows that γ_s must, for some $s \in (0, 1)$, contain some point $p \in \sigma$. Certainly $p \notin B_1 \cup B_2$. Moreover, $p \neq 0 = X(w)$ since w is a good parameter. Thus, $p \in I$. This completes the proof of the lemma. □

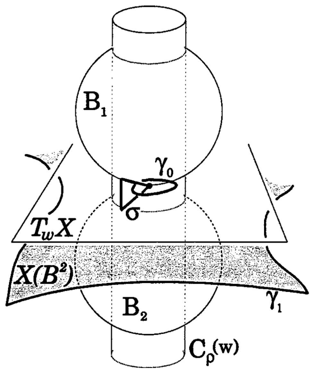

Fig. 1 Proof of the stretching property: tangent balls B_1, B_2 at $X(w)$, and the curves σ, γ_0, γ_1

Remark. A version of this lemma holds also for closed compact surfaces with global radius of curvature bounded below, see Sect. 6.3.

Using the above lemma, one easily proves that the normal to X has to change in a Lipschitz continuous way on the set of good parameters: since there are many points of X on little segments perpendicular to $T_w X$, the normal at w' close to w cannot rotate too much, for otherwise some points of the surface would belong to the forbidden balls associated to w'. Here is a quantitative statement of this observation.

Lemma 3.2 *Let $\triangle[X] \geq \theta > 0$. If $w, w' \in \mathbb{B}^2$ are good parameters such that*

$$|X(w) - X(w')| < \min(\theta, d(w)/2)$$

and $\alpha(w, w') \in [0, \frac{\pi}{2}]$ is the angle between the normal directions at w and w', then

$$\alpha(w, w') \leq \frac{5\pi}{\theta}|X(w) - X(w')|. \tag{3.1}$$

Proof. As before, suppose without loss of generality that

$$X(w) = 0, \qquad n(w) = (0, 0, 1).$$

Let

$$\begin{aligned} q_{1,2} &= X(w') \pm \theta n(w'), \\ p_{1,2} &= \pi_w(q_{1,2}), \\ p_0 &= \pi_w(X(w')), \end{aligned}$$

where π_w is the orthogonal projection onto the tangent plane to X at $X(w)$. One can assume that $\mathrm{dist}\,(p_1, 0) \leq \mathrm{dist}\,(p_2, 0)$. Set $r :\ = \mathrm{dist}\,(p_0, 0)$; obviously $r \leq |X(w) - X(w')|$ and $0 < r \leq \theta$. Pick $\lambda \geq 0$ such that

$$\mathrm{dist}\,(p_0, p_1) = \mathrm{dist}\,(p_0, p_2) = \lambda r. \tag{3.2}$$

We have $\sin\alpha(w, w') = \lambda r/\theta$. Let $d = \sqrt{\theta^2 - \lambda^2 r^2}$.

Now, if $\lambda \leq 10$, then

$$\frac{2}{\pi}\alpha(w, w') \leq \sin\alpha(w, w') \leq 10r/\theta \leq 10|X(w) - X(w')|/\theta,$$

so that the desired inequality holds.

We shall show that the assumption $\lambda > 10$ leads to a contradiction. Since $\mathrm{dist}\,(p_1, 0) \leq \mathrm{dist}\,(p_2, 0)$, we obtain from (3.2)

$$\mathrm{dist}\,(p_1, 0) \leq r\sqrt{\lambda^2 + 1} \tag{3.3}$$

(the equality holds when $\mathrm{dist}\,(p_1, 0) = \mathrm{dist}\,(p_2, 0)$). Let p_3 be that point of the segment $[p_1, 0]$ which belongs to $C_r(w)$. By the previous lemma, there exists a point

$$q_3 \in X(\mathbb{B}^2) \cap C_r(w) \cap B_{2r}(X(w))$$

such that $\pi_w(q_3) = p_3$. We shall show that for $\lambda > 10$ the distance $t := \mathrm{dist}\,(q_1, q_3)$ is smaller than θ, which contradicts the bound $\triangle[X] \geq \theta$.

Let $h' = \mathrm{dist}\,(X(w'), T_0X)$ and $h'' = \mathrm{dist}\,(q_3, p_3)$. Since there are no points of the surface X in $B_\theta(0,0,\theta) \cup B_\theta(0,0,-\theta)$, we obtain

$$\max(h', h'') \leq \theta - \sqrt{\theta^2 - r^2} \leq \frac{r^2}{\theta}.$$

Thus, the difference of z-components of q_1 and q_3 does not exceed $d + h' + h'' \leq d + 2r^2/\theta$. We also have

$$\mathrm{dist}\,(p_1, p_3) \leq r(\sqrt{\lambda^2+1} - 1) \leq r\left(\lambda - \frac{1}{2}\right)$$

(the last inequality holds for all $\lambda \geq \frac{3}{4}$). Hence,

$$\begin{aligned} t^2 &\leq \left(d + 2\frac{r^2}{\theta}\right)^2 + r^2\left(\lambda - \frac{1}{2}\right)^2 \\ &= \theta^2 + 4d\frac{r^2}{\theta} + 4\frac{r^4}{\theta^2} - r^2\lambda + \frac{r^2}{4} \\ &\leq \theta^2 + (9-\lambda)r^2 \qquad \text{as } d \leq \theta \text{ and } r \leq \theta \\ &\leq \theta^2 - r^2\,. \end{aligned}$$

Thus, q_3 is a point of X in the forbidden ball $B_\theta(q_1)$, a contradiction. □

The estimate of the last lemma is uniform, and the set of good parameters is dense. Thus, we immediately obtain the following.

Corollary 3.3 *The normal direction has a continuous extension to all $w \in \mathbb{B}^2$ and the estimate (3.1) holds for all w, w' such that $|X(w) - X(w')| \leq \min\big(\theta, d(w)/2\big)$.*

Since now we can speak of an affine normal line $\ell(w)$ at every point $X(w)$, $w \in \mathbb{B}^2$, we can associate to *each* point on the surface a pair of open balls of radius θ touching the surface without intersecting it:

Corollary 3.4 *Let $w \in \mathbb{B}^2$. Then*

$$X(\overline{\mathbb{B}}^2) \cap B_1 = X(\overline{\mathbb{B}}^2) \cap B_2 = \emptyset$$

for the two open balls B_1, B_2 centered on the normal line $\ell(w)$ with radius θ, and touching each other at $X(w)$.

Proof. If $w \in G$ this was noted already as a simple consequence of Definition 2.1 in Sect. 2. If $w \in \mathbb{B}^2 \setminus G$, and if we assume that there is a point $X(\tilde{w})$ contained in say B_1, then we derive a contradiction as follows: Since G is a dense subset of $\mathbb{B}^2$ we can find a sequence of parameters $w_j \in G$ converging to w. By the continuity of X on $\mathbb{B}^2$ and by Lemma 3.2 we obtain convergence of the normal directions $\ell(w_j)$ to $\ell(w)$ with respect to the Hausdorff distance. Associated with $\ell(w_j)$ we obtain a sequence of pairs of balls B_1^j, B_2^j of radius θ centered on $\ell(w_j)$

and touching each other in $X(w_j)$. Consequently, we obtain (relabelling $B_2^j \mapsto B_1^j$ if necessary) that $B_1^j \to B_1$ with respect to the Hausdorff distance. Hence $X(\tilde{w}) \in B_1^j$ for j sufficiently large, contradicting the fact that $B_1^j \cap X(\mathbb{B}^2) = \emptyset$, since B_1^j corresponds to a good parameter $w_j \in G$. □

4 Continuity of the normal at the boundary

From now on we assume that X is an admissible mapping, that is, $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$, with the additional property that it has a rectifiable boundary contour $X(\partial\mathbb{B}^2)$ and with $\triangle[X] \geq \theta > 0$. Moreover we suppose that the global radius of curvature $\triangle[X|_{\partial\mathbb{B}^2}]$ of the curve $X(\partial\mathbb{B}^2)$ (as defined in [15, p. 35]) is bounded below by θ. Note carefully that from now on $\triangle[\cdot]$ is used to denote two closely related but formally different notions. We always distinguish the argument in brackets, to avoid misunderstanding.

Theorem 4.1 *Let $w \in \partial\mathbb{B}^2$. If $(w_j)_{j=1,2,\ldots} \subset G \subset \mathbb{B}^2$ is a sequence of good parameters such that $w_j \to w$ as $j \to \infty$ and the normal vectors*

$$n(w_j) := \frac{X_u \wedge X_v(w_j)}{|X_u \wedge X_v(w_j)|} \xrightarrow{j\to\infty} \nu \in \mathbb{S}^2,$$

then for every good parameter $w' \in \mathbb{B}^2$ such that $|X(w') - X(w)| \leq \theta/100$ we have

$$\alpha(w, w') \leq \frac{100}{\theta}|X(w) - X(w')|, \tag{4.1}$$

where $\alpha(w, w') \in [0, \frac{\pi}{2}]$ is the angle between the affine normal line $\ell(w')$ and the line $\ell(w) := \{X(w) + t\nu \mid t \in \mathbb{R}\}$. In particular, $\ell(w)$ does not depend on the choice of the sequence (w_j).

Let us postpone the proof of this theorem for a moment, and note the following corollary which follows (as before) from the density of good parameters. We stick to the notation introduced above.

Corollary 4.2 *The normal direction (and therefore the affine tangent plane $T_w X$) has a continuous extension to all $w \in \overline{\mathbb{B}}^2$ and the estimate*

$$\alpha(w, w') \leq \frac{500}{\theta}|X(w) - X(w')| \tag{4.2}$$

holds for all $w, w' \in \overline{\mathbb{B}}^2$ such that $|X(w) - X(w')| \leq \theta/400$.

Moreover, for all $w \in \overline{\mathbb{B}}^2$ we have

$$X(\overline{\mathbb{B}}^2) \cap B_1 = X(\overline{\mathbb{B}}^2) \cap B_2 = \emptyset \tag{4.3}$$

for the two open balls B_1, B_2 centered on the normal line $\ell(w)$ with radius θ, and touching each other at $X(w)$.

Remark. B_1 and B_2 are referred to as the *boundary touching balls* in the sequel.

Proof of Corollary 4.2. If one of the points $X(w)$, $X(w')$ belongs to the boundary curve $X(\partial\mathbb{B}^2)$, we can apply Theorem 4.1 directly. Assume now that $w, w' \in \mathbb{B}^2$. If $|X(w) - X(w')| < \min(\theta, d(w)/2)$, then the desired estimate follows from Lemma 3.2.

Thus, let us suppose that $|X(w) - X(w')| \geq \min(\theta, d(w)/2)$. Then, since $|X(w) - X(w')| \leq \theta/400$, we have

$$d(w) = 2\min(\theta, d(w)/2) \leq \theta/200.$$

Fix some number $\kappa > 1$ and pick a point $w'' \in \partial\mathbb{B}^2$ such that $d(w) = |X(w) - X(w'')|$. Since $|X(w') - X(w'')| \leq \frac{\theta}{400} + \frac{\theta}{200} < \theta/100$, Theorem 4.1 and triangle inequality yield

$$\alpha(w, w') \leq \alpha(w, w'') + \alpha(w'', w') \leq \frac{100}{\theta}(d(w) + |X(w'') - X(w')|). \quad (4.4)$$

Now, consider two cases.
Case 1. If $|X(w'') - X(w')| \leq \kappa d(w)$, then the right-hand side of (4.4) does not exceed

$$\frac{100(\kappa+1)}{\theta} d(w) \leq \frac{200(\kappa+1)}{\theta} |X(w) - X(w')|.$$

Case 2. If $|X(w'') - X(w')| > \kappa d(w)$, then

$$|X(w) - X(w')| \geq |X(w'') - X(w')| - d(w) \geq (1 - \kappa^{-1})|X(w'') - X(w')|,$$

and hence

$$\begin{aligned} d(w) + |X(w'') - X(w')| &< (1 + \kappa^{-1})|X(w'') - X(w')| \\ &\leq \frac{\kappa+1}{\kappa-1} |X(w') - X(w)|. \end{aligned}$$

Choosing $\kappa = 3/2$, in either case we estimate the right-hand side of (4.4) by $\frac{500}{\theta}|X(w') - X(w)|$, and inequality (4.2) follows.

To prove the second statement (i.e. the existence of excluded touching balls also at the boundary), one mimics the proof of Corollary 3.4. □

Proof of Theorem 4.1. Since the whole proof is rather long (yet elementary), we describe roughly the main idea. First, we prove that if the surface contains a point $X(w')$, $w' \in \mathbb{B}^2$ close to $X(w)$, then it contains also lots of other points $X(w'')$ lying very close to some half-circle centered at $X(w)$, perpendicular to ν and of radius approximately $|X(w) - X(w')|$. This is the boundary counterpart of the stretching property from the previous section; as before, the argument is a topological one. Next, we show that the normal direction at $X(w')$ must be close to ν, for otherwise the excluded balls associated to w' would contain one of the points $X(w'')$ constructed in the first step of the proof. This part of the argument is completely geometric.

With no loss of generality we assume that $X(w) = (0,0,0) \in \mathbb{R}^3$ and $\nu = (0,0,1) \in \mathbb{S}^2$. The pairs of forbidden balls

$$U_j := B_\theta(X(w_j) + \theta n(w_j)) \cup B_\theta(X(w_j) - \theta n(w_j))$$

converge in the Hausdorff distance, as $j \to \infty$, to

$$U := B_\theta(0,0,\theta) \cup B_\theta(0,0,-\theta)\,.$$

It is clear $X(\mathbb{B}^2) \cap U = \emptyset$, for otherwise there would be points of the surface X in U_j for j sufficiently large. Thus, if $\Gamma\colon [0,L] \to \mathbb{R}^3$ is the arc-length parametrization of $X(\partial\mathbb{B}^2)$ such that $\Gamma(0) = X(w)$, then $\Gamma'(0) = (a,b,0)$ for some $a, b \in \mathbb{R}$. Without loss of generality suppose that $\Gamma'(0) = (1,0,0)$ and set

$$V = \bigcup_{\alpha \in [0,2\pi]} B_\theta(0, \theta\cos\alpha, \theta\sin\alpha)\,.$$

Since $\triangle[\Gamma] \geq \theta$, by [26, Thm. 1(iv)(a) and Lemma 2] we infer that $X(\partial\mathbb{B}^2) \subset \mathbb{R}^3 \setminus V$. As before, let

$$C_\varrho(w) = \{p \in \mathbb{R}^3 \mid \operatorname{dist}(p, \ell(w)) = \varrho\}\,.$$

Set

$$h(\varrho) = \theta - \sqrt{\theta^2 - \varrho^2} \qquad \text{for } \varrho \in [0,\theta]. \tag{4.5}$$

Consider two narrow, rectangular patches Σ_ϱ^+ and Σ_ϱ^- lying on the cylinder $C_\varrho(w)$, which are defined by

$$\Sigma_\varrho^\pm = \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = \varrho^2,\ |x_3| \leq h(\varrho),\ \pm x_2 \geq \varrho^2/2\theta\right\}. \tag{4.6}$$

Elementary computations show that

$$\Sigma_\varrho^+ \cup \Sigma_\varrho^- \subset (\mathbb{R}^3 \setminus U) \cap V.$$

(In fact, the horizontal edges of $\Sigma_\varrho^\pm$ touch the boundary of U.) Let $\sigma_\varrho^\pm$ denote two circular arcs along which the horizontal plane $\{x_3 = 0\}$ intersects $\Sigma_\varrho^\pm$, respectively. Using (4.6) one easily checks that the length

$$|\sigma_\varrho^+| = |\sigma_\varrho^-| = (\pi - 2\beta_\varrho)\varrho,$$

where $\beta_\varrho = \arctan \varrho/\sqrt{4\theta^2 - \varrho^2}$. Instead of this formula, we just use the estimate

$$\beta_\varrho \leq \tan\beta_\varrho = \frac{\varrho}{\sqrt{4\theta^2 - \varrho^2}} \leq \frac{\varrho}{\theta} \qquad \text{for } \varrho \leq \theta.$$

In particular, for $\varrho \leq \theta/100$ we have $2\beta_\varrho \leq \pi/100$, and therefore

$$|\sigma_\varrho^\pm| \geq 0.99\pi\varrho \qquad \text{for } \varrho \leq \theta/100. \tag{4.7}$$

Now, $\Sigma_\varrho^\pm = \sigma_\varrho^\pm \times I_\varrho$, where I_ϱ denotes the interval $[-h(\varrho), h(\varrho)]$. We shall need the following

Claim 1. *One of the following two conditions is satisfied:*

(i) *For each $p \in \sigma_\varrho^+$ there is point in $X(\mathbb{B}^2) \cap (\{p\} \times I_\varrho)$;*
(ii) *For each $p \in \sigma_\varrho^-$ there is point in $X(\mathbb{B}^2) \cap (\{p\} \times I_\varrho)$.*

In other words, if π_w is the orthogonal projection onto $\{x_3 = 0\}$, then $\pi_w(X(\mathbb{B}^2) \cap \Sigma_\varrho^+) = \sigma_\varrho^+$ or $\pi_w(X(\mathbb{B}^2) \cap \Sigma_\varrho^-) = \sigma_\varrho^-$

Proof of Claim 1. Assume both (i) and (ii) are false. Then, there exist two points $p_1 \in \sigma_\varrho^+$ and $p_2 \in \sigma_\varrho^-$ such that two segments, $J_1 := \{p_1\} \times I_\varrho$ and $J_2 := \{p_2\} \times I_\varrho$, contain no points of the surface X. Consider the closed curve γ which consists of J_1 and J_2, and two other horizontal segments J_3, J_4 that join the endpoints of J_1 and J_2 (i.e., each of J_3, J_4 is contained in a plane $\{x_3 = \text{const}\}$, in one of the balls $B_\theta(0, 0, \pm\theta)$). We have

$$\gamma = J_1 \cup J_2 \cup J_3 \cup J_4 \subset V.$$

Moreover, the curves γ and $X(\partial\mathbb{B}^2)$ are linked.

Now, consider a homotopy $(\varphi_t)_{t\in[0,1]}$ which deforms the whole boundary curve $\varphi_0 = X(\partial\mathbb{B}^2)$ to a small loop φ_1 located near zero, in $B_s(0) \setminus U$, where $s = h(\varrho)/4$. A small arc of $X(\partial\mathbb{B}^2)$ that contains $0 = X(w)$ in its interior is kept fixed, and the remaining portion of $X(\partial\mathbb{B}^2)$ is being deformed, using the composition of X with a suitable homotopy in the domain $\mathbb{B}^2$. As $h(\varrho) \le \varrho$ for all $\varrho \le \theta$, we have $\operatorname{dist}(\gamma, 0) \ge h(\varrho)$; thus it is clear that φ_1 and γ are *not* linked. Hence, one of the intermediate curves φ_t must intersect γ, and this is possible only when there is a point of $X(\mathbb{B}^2)$ in $J_1 = \{p_1\} \times I_\varrho$ or in $J_2 = \{p_2\} \times I_\varrho$, since all the remaining points of γ belong to the forbidden balls U. This contradiction ends the proof of Claim 1.

Choose an arbitrary good parameter w' with $|X(w') - X(w)| \le \theta/100$. Let

$$q_{1,2} = X(w') \pm \theta \frac{X_u \wedge X_v(w')}{|X_u \wedge X_v(w')|}$$

denote the centers of forbidden balls corresponding to w', and let

$$p_{1,2} = \pi_w(q_{1,2}), \qquad p_0 = \pi_w(X(w')).$$

From now on we fix $\varrho = \operatorname{dist}(p_0, 0)$ and suppose w.l.o.g. that for this particular ϱ condition (i) holds. Take $\lambda \ge 0$ such that $\operatorname{dist}(p_0, p_1) = \operatorname{dist}(p_0, p_2) = \lambda\varrho$. We have

$$\sin\alpha(w, w') = \frac{\lambda\varrho}{\theta} \le \frac{\lambda}{\theta}|X(w) - X(w')|.$$

If $\lambda \le 50$ there is nothing more to prove. Thus, assume that $\lambda > 50$. We shall show that this leads to a contradiction.

Claim 2. *If $\lambda > 50$ then there exists a point $p_3 \in \sigma_\varrho^+$ such that*

$$\min(\operatorname{dist}(p_1, p_3), \operatorname{dist}(p_2, p_3)) \le \left(\lambda - \frac{1}{8}\right)\varrho.$$

Proof of Claim 2. Let c_ϱ denote the circle in $\{x_3 = 0\}$ which contains the arcs $\sigma_\varrho^\pm$. We have $p_0 \in c_\varrho$. Let $\epsilon_0 \in [0, \frac{\pi}{2}]$ be the angle between $[p_1, p_2]$ and the tangent line to c_ϱ at p_0. We distinguish two cases.

Case 1. $\epsilon_0 \le \pi/12$. Introduce complex coordinates in the $\{x_3 = 0\}$ plane and let

$$p_3' = \exp(i\pi/2)p_0, \qquad p_3'' = \exp(-i\pi/2)p_0.$$

Without loss of generality suppose that the angle between $[p_0, p_1]$ and $[p_0, p_3']$ is equal to $\pi/4 \pm \epsilon_0$ (otherwise exchange the roles of p_3' and p_3''). Let $d = \operatorname{dist}(p_1, p_3')$. By the law of cosines,

$$\begin{aligned} d^2 &= \lambda^2\varrho^2 + 2\varrho^2 - 2\lambda\varrho^2\sqrt{2}\cos\left(\tfrac{\pi}{4} \pm \epsilon_0\right) \\ &\le \varrho^2(\lambda^2 + 2 - 2\lambda\sqrt{2}\cos\tfrac{\pi}{3}) \\ &\le \varrho^2\left(\lambda - \frac{1}{4}\right)^2 \qquad \text{for every } \lambda > 50. \end{aligned}$$

Similarly,

$$\operatorname{dist}(p_2, p_3'') \le \varrho\left(\lambda - \frac{1}{4}\right).$$

If p_3' or p_3'' belongs to σ_ϱ^+, we are done. If this is not the case, then, in view of (4.7), the distance of p_3' to an endpoint of σ_ϱ^+ is smaller than $\pi\varrho/100$ (and the same holds for p_3'' and the other endpoint of σ_ϱ^+). Claim 2 holds then with p_3 equal to any point of σ_ϱ^+ which is sufficiently close to one of its endpoints.

Case 2. $\epsilon_0 \ge \pi/12$. Without loss of generality assume that $\operatorname{dist}(p_1, 0) \le \operatorname{dist}(p_2, 0)$. By the law of cosines,

$$\begin{aligned} d^2 :&= \operatorname{dist}(p_1, 0)^2 = \lambda^2\varrho^2 + \varrho^2 - 2\lambda\varrho^2\cos\left(\tfrac{\pi}{2} - \epsilon_0\right) \\ &= \varrho^2(\lambda^2 + 1 - 2\lambda\sin\epsilon_0) \\ &\le \varrho^2\left(\lambda^2 - \frac{\lambda}{2} + 1\right). \end{aligned}$$

On the other hand, $\operatorname{dist}(p_1, 0) \ge (\lambda - 1)\varrho$ by the triangle inequality. Thus,

$$(\lambda - 1)^2 \le \frac{d^2}{\varrho^2} \le \lambda^2 - \frac{\lambda}{2} + 1. \tag{4.8}$$

Now, let p_3' and p_3'' be the points of c_ϱ at which the tangents c_ϱ passing through p_1 intersect c_ϱ. By (4.8), we have

$$\begin{aligned} \operatorname{dist}^2(p_1, p_3') = \operatorname{dist}^2(p_1, p_3'') &= d^2 - \varrho^2 \\ &\le \varrho^2\left(\lambda^2 - \frac{\lambda}{2}\right) \le \varrho^2\left(\lambda - \frac{1}{4}\right)^2. \end{aligned} \tag{4.9}$$

Thus, if p_3' or p_3'' belongs to σ_ϱ^+, then we are done. Otherwise, note that the shorter of two arcs $p_3'p_3''$ has the length equal to $(\pi - 2\delta)\varrho$, where $\sin\delta = \varrho/d$. Using (4.8), we easily estimate

$$|\text{arc } p_3'p_3''| = (\pi - 2\delta)\varrho \ge \pi(1 - \sin\delta)\varrho \ge \pi\left(1 - \frac{1}{\lambda - 1}\right)\varrho.$$

Since $\lambda > 50$, we can combine this information and (4.7) to obtain

$$\min(\operatorname{dist}(p_3', \sigma_\varrho^+), \operatorname{dist}(p_3'', \sigma_\varrho^+)) \leq \frac{\pi\varrho}{40} \leq \frac{\varrho}{10}.$$

Thus, invoking (4.9) and the triangle inequality, we conclude the whole proof of Claim 2.

The rest of proof of Theorem 4.1 resembles the proof of the interior continuity of the normal direction. Assume that $\operatorname{dist}(p_1, p_3) \leq \varrho(\lambda - \frac{1}{8})$, where $p_3 \in \sigma_\varrho^+$. Let $q_3 \in (\{p_3\} \times I_\varrho) \cap X(\mathbb{B}^2)$. Set $s = \sqrt{\theta^2 - \lambda^2\varrho^2}$; since the surface cannot penetrate the forbidden balls U, the difference of the 'vertical' coordinates of q_3 and $q_1 = X(w') + \theta n(w')$ does not exceed $s + 2h(\varrho) \leq s + 2\varrho^2/\theta$. Hence,

$$\begin{aligned}
\operatorname{dist}(q_1, q_3)^2 &\leq \left(s + 2\frac{\varrho^2}{\theta}\right)^2 + \varrho^2\left(\lambda - \frac{1}{8}\right)^2 \\
&= \theta^2 + 4s\frac{\varrho^2}{\theta} + 4\frac{\varrho^4}{\theta^2} - \frac{\lambda\varrho^2}{4} + \frac{\varrho^2}{64} \\
&\leq \theta^2 + \left(9 - \frac{\lambda}{4}\right)\varrho^2 \qquad \text{as } s \leq \theta \text{ and } \varrho \leq \theta \\
&< \theta^2 \qquad \text{when } \lambda > 50.
\end{aligned}$$

Therefore, $q_3 \in B_\theta(q_1) \cap X(\mathbb{B}^2)$, a contradiction. □

5 Structure of the image

In this section we prove that an admissible surface $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$ with $\triangle[X] \geq \theta > 0$ and with sufficiently nice boundary behaviour parametrizes a $C^{1,1}$-manifold with boundary. We also obtain some additional, more precise information concerning local properties of this manifold.

Throughout this section we use the following notation. If $p = X(w) \in X(\overline{\mathbb{B}}^2)$, $\pi_w \colon \mathbb{R}^3 \to T_w X$ is the orthogonal projection, and $\ell(w)$ denotes the affine normal line passing through p, then

$$V_\varrho(p) \colon = \{q \in \mathbb{R}^3 \mid \operatorname{dist}(q, \ell(w)) < \varrho,\ |q - \pi_w(q)| < \varrho\}$$

is a solid open cylinder with axis parallel to $\ell(w)$, centered at $p = X(w)$, with radius $\varrho > 0$ and height 2ϱ.

Theorem 5.1 *Let $\triangle[X] \geq \theta > 0$ and let $w \in G \subset \mathbb{B}^2$ be a good parameter such that*

$$\operatorname{dist}(X(w), X(\partial\mathbb{B}^2)) > 2\sigma\theta\,,$$

where $\sigma \in (0, 1/100]$ can be chosen at will. Then $X(\mathbb{B}^2) \cap V_{\sigma\theta}(X(w))$ is a graph of a function $g \in C^{1,1}(B^2_{\sigma\theta}(0), \mathbb{R})$ with $\|g\|_{C^{1,1}} \leq C/\theta$, $\operatorname{Lip} g \leq 1$, where C is some absolute constant.

Thus, loosely speaking, a portion of the surface contained in a cylinder of size comparable to θ is a graph of a $C^{1,1}$ function. The norm of this function is estimated inversely proportional to θ.

Before passing to the proof, we state the following.

Remark. In fact, the assumption $\triangle[X] \geq \theta$ is not applied directly in the proof. What matters is the existence of excluded touching balls for every point in $X(\overline{\mathbb{B}}^2)$, as given in Corollary 3.4, and Lipschitz continuity (w.r.t. to distances measured in the image) of the normal direction $\ell(w)$. Thus, Theorem 5.1 applies to any continuous surface for which the excluded balls exist at every point in the image, and the line joining their centers varies in a Lipschitz continuous way. The original parametrization is not really important here.

Proof. We can assume

$$X(w) = 0 \in \mathbb{R}^3, \qquad n(w) = (0, 0, 1).$$

Since X has the ϱ-stretching property for $\varrho :\ = \sigma\theta$, there is a point of $X(\mathbb{B}^2) \cap B_\varrho(0)$ on each line ℓ' parallel to $\ell(w)$ (= the x_3-axis) such that dist $(\ell', \ell(w)) < \sigma\theta$. In fact, there is at most *one* such point (otherwise it is easy to use Lemma 3.2 to obtain a contradiction).

So, if $z = (x, y) \in B^2_{\sigma\theta}(0) \subset T_w X$, and $p \in X(\mathbb{B}^2) \cap V_{\sigma\theta}(0)$ is the unique point such that $\pi_w(p) = z$, then we set $g(z) := p_3$. In other words, $g = \pi_3 \circ (\pi_w |_{X(\mathbb{B}^2)\cap V_{\sigma\rho}(0)})^{-1}$, where π_3 is the projection of $\mathbb{R}^3$ onto the x_3-axis.

Step 1. Since $X(\mathbb{B}^2)$ is contained in the complement of excluded balls at $X(w)$, we have

$$|g(z)| \leq |z|^2, \quad z \in B^2_{\sigma\theta}(0),$$

and hence

$$g(z) = g(0) + Dg(0)z + o(|z|) \quad \text{as } |z| \to 0,$$

with $Dg(0) = (0, 0)$. At any other point $z \in B^2_{\sigma\theta}(0)$ a similar argument works: the graph of g is trapped between two mutually tangent balls of radius θ that touch each other at $(z, g(z)) \in \mathbb{R}^3$. This implies differentiability of g everywhere.

Step 2. The vector $(Dg(z), 1)$ is parallel to the normal direction to X at w' when $z = \pi_w(X(w'))$. By Lemma 3.2, for each w' such that $X(w') \in V_\varrho(0)$ we have

$$\alpha(w, w') \leq \frac{5\pi}{\theta}|X(w) - X(w')| \leq \frac{5\pi}{\theta} \cdot 2\sigma\theta < \frac{\pi}{4}\,.$$

Since $\tan\alpha(w, w') = |Dg(z)|$, we have $|Dg(z)| \leq 1$ everywhere in $B^2_{\sigma\theta}(0)$. Thus, g is Lipschitz with Lipschitz constant 1.

Step 3. Fix two points $z_1, z_2 \in B^2_{\sigma\theta}(0)$ and set $a = g_x(z_1)$, $b = g_y(z_1)$, $c = g_x(z_2)$, $d = g_y(z_2)$. The angle α between the normal directions to $X(\mathbb{B}^2)$ at $(z_i, g(z_i))$, $i = 1, 2$, satisfies

$$\begin{aligned} \sin^2\alpha \quad &= \quad \frac{(a-c)^2 + (b-d)^2 + (ad-bc)^2}{(1+a^2+b^2)(1+c^2+d^2)} \\ &\overset{\text{(Step 2)}}{\geq} \frac{(a-c)^2 + (b-d)^2}{4} = \frac{|Dg(z_1) - Dg(z_2)|^2}{4}\,. \end{aligned} \tag{5.1}$$

Suppose now that

$$|Dg(z_1) - Dg(z_2)| \geq K|z_1 - z_2| \tag{5.2}$$

for some constant K. As g has Lipschitz constant 1, this yields

$$|Dg(z_1) - Dg(z_2)| \geq \frac{K}{2}\big(|z_1 - z_2| + |g(z_1) - g(z_2)|\big). \tag{5.3}$$

Combining inequalities (5.1) and (5.3) with the estimate of $\sin\alpha$ resulting from Lemma 3.2, we obtain $K/4 \leq 5\pi/\theta$. Hence Dg is Lipschitz with constant $20\pi/\theta$. □

There is also a boundary counterpart of the last theorem:

Theorem 5.2 *Let $p \in X(\partial\mathbb{B}^2)$, where $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$ is an admissible surface with $\triangle[X] \geq \theta$ and $\triangle[X\big|_{\partial\mathbb{B}^2}] \geq \theta$ for some $\theta > 0$. Assume that $X\big|_{\partial\mathbb{B}^2} \colon \partial\mathbb{B}^2 \to X(\partial\mathbb{B}^2)$ is weakly monotone. Then, there exists a function $g \in C^{1,1}(B^2_{\theta/300}(0))$ such that $\|g\|_{C^{1,1}} \leq C/\theta$,* Lip $g \leq 1$*,*

$$X(\overline{\mathbb{B}}^2) \cap V_{\theta/300}(p) = \operatorname{Graph}\,(g\big|\Omega^+_{bd})\,,$$

where $B^2_{\theta/300}(0)$ is a disk in T_wX and $\Omega^+_{bd} = \{(x, y) \in B^2_{\theta/300}(0) \,:\, y \geq \psi(x)\}$ for some function ψ of class $C^{1,1}(\mathbb{R})$ with $\psi(0) = \psi'(0) = 0$ and $\|\psi\|_{C^{1,1}} \leq C/\theta$.

Proof. Fix $p = X(w) \in X(\partial\mathbb{B}^2)$. Assume w.l.o.g. that $p = 0 \in \mathbb{R}^3$ and that the limit of normal directions to X at p coincides with the x_3 axis. Let

$$\pi_w \colon \mathbb{R}^3 \to T_wX \equiv \{x_3 = 0\}$$

be the orthogonal projection onto T_wX. Set $\varrho = \theta/300$.

Step 1: definition of ψ. Assume w.l.o.g. that $X(\partial\mathbb{B}^2)$ has an arc-length parametrization $\Gamma\colon [-L/2, L/2] \to \mathbb{R}^3$ with $\Gamma(0) = p$, $\Gamma'(0) = (1, 0, 0)$. Let $\gamma = \pi_w[X(\partial\mathbb{B}^2) \cap V_\varrho(p)]$; the curve $\gamma \subset B^2_\varrho(0) \subset \mathbb{R}^2 \equiv T_wX$ is parametrized by $(\Gamma_1, \Gamma_2, 0)$. Since

$$|\Gamma'(t) - \Gamma'(s)| \leq \frac{1}{\theta}|\Gamma(t) - \Gamma(s)|,$$

we obtain in particular $1 \geq \Gamma_1'(t) \geq 99/100$ and $|\Gamma_2'(t)| \leq 1/100$ for all t such that $\Gamma(t) \in V_\varrho(p)$. Inverting Γ_1, we see that $\gamma = \operatorname{Graph}\psi$ for some function ψ of one variable, $\psi(0) = \psi'(0) = 0$, $\|\psi\|_{C^{1,1}} \leq C/\theta$. (See e.g. the appendix of Norton's paper [24] for a version of the Implicit and Inverse Function theorem in Hölder classes.)

Step 2: we claim that $\pi_w\big|_{X(\mathbb{B}^2)\cap V_\varrho(p)}$ is 1–1. This follows easily from Theorem 4.1. Indeed, let $X(w') = (x_1, x_2, x_3) \in V_\varrho(p)$. Take the 'vertical' segment

$$I = \{(x_1, x_2, t) :\ |t| \leq \theta - (\theta^2 - x_1^2 - x_2^2)^{1/2}\}$$

which passes through $X(w')$ and has both endpoints on the forbidden balls associated to $p = X(w)$. An elementary computation shows that $I \setminus \{X(w')\}$ is contained in (the interior of) forbidden balls associated to $X(w')$. Thus, $X(w')$ is the only point in the image of X intersected with I; the claim follows.

Step 3. We claim that $\pi_w\big[X(\mathbb{B}^2)\cap V_\varrho(p)\big]$ is equal either to

$$\Omega^+ = \{(x, y) \in B^2_\varrho(0) \mid y > \psi(x)\}$$

or to $\Omega^- = \{(x, y) \in B^2_\varrho(0) \mid y < \psi(x)\}$ (i.e. a portion of the surface near p projects onto *only* one side of γ).

Assume w.l.o.g. that $\Omega^+ \cap \pi_w[X(\mathbb{B}^2)\cap V_\varrho(p)]$ is nonempty. Then $\Omega^+ \subset \pi_w[X(\mathbb{B}^2)\cap V_\varrho(p)]$, for otherwise we could find either

(A) a piece γ_1 of $X(\partial\mathbb{B}^2)$ entering the cylinder $V_\varrho(p)$ in such a way that $\Omega^+ \supset \pi_w(\gamma_1) \neq \operatorname{graph}\psi$,

or

(B) a parameter $w' \in \mathbb{B}^2$ such that $q = X(w')$ belongs to

$$\Omega^+ \cap \partial(\pi_w(X(\mathbb{B}^2)\cap V_\varrho(p))).$$

However, (A) would contradict the assumption $\triangle[X|_{\partial\mathbb{B}^2}] \geq \theta > 0$ and the description of $X(\partial\mathbb{B}^2)$ obtained in Step 1 above. (B) implies that the normal at q must be perpendicular to the x_3-axis, whereas $|X(w) - X(w')| < 2\varrho$, which is a contradiction to (4.1).

If $\Omega^- \cap \pi_w[X(\mathbb{B}^2)\cap V_\varrho(p)] = \emptyset$, then we are done. If this is not the case, we can assume — repeating the reasoning above — that $\Omega^- \subset \pi_w[X(\mathbb{B}^2)\cap V_\varrho(p)]$. We shall show that this assumption leads to a contradiction.

To this end, we first note the following:

$$\text{If } w' \in \operatorname{int}\mathbb{B}^2, \text{ then } X(w') \notin X(\partial\mathbb{B}^2)\cap V_\varrho(p). \tag{5.4}$$

We prove this by contradiction. Assume that $X(w') = q \in X(\partial\mathbb{B}^2)\cap V_\varrho(p)$. Pick $\delta, r > 0$ such that

$$B_r(q) \subset V_\varrho(p), \tag{5.5}$$

$$\delta < (1 - |w'|)/2 \quad \text{and} \quad X(B_\delta(w')) \subset B_r(q). \tag{5.6}$$

Moreover, $X(B_\delta(w'))$ must be contained in the complement of the forbidden ‘double hose’ formed by the boundary touching balls at $X(w')$ and nearby points of $X(\partial\mathbb{B}^2)$.

Now, two cases are possible.

(C_1) $X(B_\delta(w'))$ projects both onto Ω^+ and Ω^-.

Then, ‘above’ Ω^+ a portion of $X(B_\delta(w'))$ must overlap with a portion of $X(B_\delta(w''))$, where $w'' \in \partial\mathbb{B}^2$ and $X(w'') = q = X(w')$. (We cannot have two different layers of the surface in $V_\varrho(p)$, see Step 2!). This is impossible, since good parameters are dense, and cannot be mapped to image points of other parameters.

(C$_2$) $X(B_\delta(w'))$ projects only to one side of graph ψ, say to Ω^-.

Then, using the proximity of normal directions at all points of $X(B_\delta(w'))$ to the normal at $q = X(w')$, we see (precisely as above) that the piece $X(B_\delta(w'))$ can have neither two different layers contained in $V_\varrho(p)$ nor a doubly covered part, which again leads to a contradiction.

Thus, (5.4) is established.

Now, take an arbitrary point $q \in X(\partial\mathbb{B}^2) \cap V_\varrho(p)$. Since both Ω^+ and Ω^- are covered by the projection of $X(\overline{\mathbb{B}}^2) \cap V_\varrho(p)$, we can use (5.4) to find *two* different parameters $w^+, w^- \in \partial\mathbb{B}^2$ such that $X(w^+) = X(w^-) = q$, and the image of a small neighbourhood of $w^\pm$ projects onto $\Omega^\pm$, respectively. As X is weakly monotone, a whole closed arc $\sigma \subset \partial\mathbb{B}^2$ joining w^+ to w^- satisfies $X(\sigma) = p$. We use uniform continuity of X to find an $\varepsilon > 0$ so small that

$$X(B_\varepsilon(\sigma) \cap \overline{\mathbb{B}}^2) \subset V_\varrho(p).$$

Now, in the interior of $B_\varepsilon(\sigma) \cap \overline{\mathbb{B}}^2$ we can easily find a curve σ' that joins two points w_1, w_2 such that

$$\pi_w(X(w_1)) \in \Omega^+, \quad \pi_w(X(w_2)) \in \Omega^-.$$

By continuity, σ' must contain a point $w_0 \in \mathbb{B}^2$ which is mapped by X to the boundary curve $X(\partial\mathbb{B}^2)$ This contradicts (5.4), and finally proves that $\Omega^- \cap \pi_w[X(\mathbb{B}^2 \cap V_\varrho(p)]$ is empty, completing Step 3 of the proof.

Step 4. From now on we assume w.l.o.g. that

$$\pi_w[X(\mathbb{B}^2) \cap V_\varrho(p)] = \Omega^+ \subset B^2_\varrho(0).$$

Then we define $g\colon \Omega^+ \to \mathbb{R}$, setting

$$g(z) = q_3 \quad \text{when } q = (q_1, q_2, q_3) \in X(\mathbb{B}^2) \cap V_\varrho(p) \text{ is such that } \pi_w(q) = z.$$

Step 5. Replacing Lemma 3.2 by Theorem 4.1 in all arguments, one proves the existence of $Dg(z)$ for all z and the estimate $|Dg(z)| \le 1$ precisely as in the previous Theorem. Lipschitz continuity of Dg and the estimate $\|g\|_{C^{1,1}(\Omega^+)} \le C/\theta$ follows (one mimicks the last part of the proof of Theorem 5.1).

Step 6. To extend g from Ω^+ to the whole disk $B^2_\varrho(0)$ without increasing too much its $C^{1,1}$-norm, we apply [13, Lemma 6.37]. □

Example. An open rotational cylinder with two hemispheres of the same radius attached at both ends shows that $C^{1,1}$-regularity is optimal for surfaces, for which pairs of excluded touching balls exist in the image: this particular surface fails to be C^2 at all points where the hemispheres meet the cylinder.

Combining both theorems, we obtain the following.

Corollary 5.3 *Assume that X satisfies the assumptions of Theorem 5.2. There exist two absolute constants $\sigma_0 > 0$, $K \ge 1$ such that for each $\sigma \le \sigma_0$ and $p \in X(\overline{\mathbb{B}}^2)$ we have*

$$K^{-1}\sigma^2\theta^2 \le \mathcal{H}^2(B_{\sigma\theta}(p) \cap X(\overline{\mathbb{B}}^2)) \le K\sigma^2\theta^2. \tag{5.7}$$

In plain words, pieces of the surface in a ball of radius $\delta \lesssim \theta$ have their area comparable to the area of a flat disk of radius δ.

Moreover, we have the following.

Lemma 5.4 *Under the assumptions of Theorem 5.2, $X(\overline{\mathbb{B}}^2)$ is homeomorphic to a closed disk (and hence orientable).*

Proof. Set $A := X(\overline{\mathbb{B}}^2)$. It follows from the proof of Theorem 5.2 that $\partial\mathbb{B}^2$ is mapped onto $\partial A \approx \mathbb{S}^1$ in a weakly monotone (= degree one) way. Thus, there exists a homotopy $H : \mathbb{S}^1 \times [0,1] \to \partial A$ such that

$$H(e^{i\theta}, 0) = X(e^{i\theta}), \quad H(e^{i\theta}, 1) = \varphi(e^{i\theta}) \qquad \text{for all } \theta \in [0, 2\pi],$$

where φ is a fixed 1–1 parametrization of ∂A.

We argue by contradiction. Assume that A is not homeomorphic to the disk. Then, A cannot be homeomorphic to the Möbius band M, as contracting $\partial\mathbb{B}^2$ to a point in $\mathbb{B}^2$ and using X to transport this homotopy to the image, we would obtain a homotopy from a point in M to a loop that circumvents ∂M once, which is impossible.

Thus, A must be some other $C^{1,1}$-surface (orientable or not) with boundary homeomorphic to $\mathbb{S}^1$. We now close the hole in A. To this end we identify $\mathbb{R}^3$ with a hyperplane $\mathbb{R}^3 \times \{0\}$ in $\mathbb{R}^4$, take another copy $\overline{U}$ of a closed disk and a map $Y : \overline{U} \to \mathbb{R}^4$, with $Y\big|_{\partial U}$ identical to the boundary values of X, to attach this disk to A along ∂A. Using H, we define Y by the formula

$$Y(w) = \begin{cases} \left(H\left(\frac{w}{|w|}, 2 - 2|w|\right), 1 - |w|\right) & \text{if } |w| \in \left[\frac{1}{2}, 1\right], \\ (2|w|\varphi(\frac{w}{|w|}), 1 - |w|) & \text{if } |w| \in \left(0, \frac{1}{2}\right), \\ (0,0,0,1) & \text{if } w = 0. \end{cases} \tag{5.8}$$

Let Σ denote the new surface thus obtained.

Glueing $\overline{U}$ and the original disk $\overline{\mathbb{B}}^2$ along their boundaries, we obtain a map $f : \mathbb{S}^2 \to \Sigma$, equal to X on one hemisphere and to Y on the other one. Since A is neither a disk nor a Möbius band, Σ is different from $\mathbb{S}^2$ and the projective plane $\mathbb{RP}^2$. Therefore, the second homotopy group $\pi_2(\Sigma)$ vanishes, see e.g. [8, p. 198], and the map f is homotopic to the constant map. Thus, the $\mathbb{Z}_2$-degree (= degree mod 2) of f, which is well defined also for non-orientable surfaces [8, pp. 102–106], should be zero.

However, it follows from the definition of f and Y that each point p in $f(V)$, $V := B_{1/2}(0)$, has its fourth coordinate $p_4 \in [\frac{1}{2}, 1]$. Since A lies in the hyperplane $\{p_4 = 0\} \subset \mathbb{R}^3$, one easily checks that $f^{-1}(p)$ is contained only in V and consists of a *single* point. Thus, the $\mathbb{Z}_2$-degree of f equals 1, a contradiction. □

The theorems of this section show how strong in fact the assumption $\triangle[X] \geq \theta > 0$ is. Even if the global curvature radius of the boundary curve $\gamma := X(\partial\mathbb{B}^2)$ is positive, γ can be badly knotted. However, if we know *in addition* that γ bounds a surface $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$ with $\triangle[X] \geq \theta > 0$, then it follows from Theorems 5.1 and 5.2 that γ cannot be knotted! This is vaguely reminiscent of the famous Fáry–Milnor theorem [22], [7, p. 402] relating curvature to topological aspects.

6 Convergence and compactness

6.1 Sets of positive reach

Following Federer [10, Sect. 4] we define the *reach* of a set $A \subset \mathbb{R}^d$ as

$$\operatorname{reach}(A) := \sup\{r \in \mathbb{R} : \forall x \in B_r(A)\ \exists_1 a \in A \text{ with } \operatorname{dist}(x, A) = |x - a|\}.$$

Hence for any $x \in B_r(A)$, $r < \operatorname{reach}(A)$, there is a unique next point $a \equiv \Pi_A(x) \in A$, such that $\operatorname{dist}(x, A) = |x - \Pi_A(x)|$. If the set $A \subset \mathbb{R}^d$ is closed, then the map $\Pi_A(.) : B_{\operatorname{reach}(A)}(A) \to A$ is continuous, cf. [10, Thm. 4.8 (4)]. Given a set $A \subset \mathbb{R}^d$ and a point $a \in A$ one can define the *tangent cone* $T_a A$ as

$$\begin{aligned} T_a A := \{v \in \mathbb{R}^d\ &: v = 0 \\ &\text{or } \forall \epsilon > 0\ \exists b \in A \cap B_\epsilon(a) \text{ such that } \left| \tfrac{b-a}{|b-a|} - \tfrac{v}{|v|} \right| < \epsilon\}, \end{aligned}$$

which reduces to the classical (linear, not affine!) tangent plane $T_a \Sigma$, if $A = \Sigma \subset \mathbb{R}^d$ is a C^1-submanifold in $\mathbb{R}^d$ (cf. [10, Rmk. 4.6]). Federer characterizes closed sets of positive reach by an inequality reflecting a uniform second order contact between the set and its tangent cone:

Theorem 6.1 [10, Thm. 4.18]
For a closed set $A \subset \mathbb{R}^d$ and $t \in (0, \infty)$ one has $\operatorname{reach}(A) \geq t$ *if and only if*

$$2 \operatorname{dist}(b - a, T_a A) \leq \frac{|b - a|^2}{t} \text{ for all } a, b \in A. \tag{6.1}$$

Returning to our setting of admissible surfaces $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$ we can now easily prove

Lemma 6.2 *Let $X \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$ with $\triangle[X] > 0$ and $\triangle[X\big|_{\partial\mathbb{B}^2}] > 0$ and $\theta > 0$. Then the following two statements are equivalent:*

(i) $\triangle[X] \geq \theta$ *and* $\triangle[X\big|_{\partial\mathbb{B}^2}] \geq \theta$,
(ii) $\operatorname{reach}(X(\overline{\mathbb{B}}^2)) \geq \theta$ *and* $\operatorname{reach}(X(\partial\mathbb{B}^2)) \geq \theta$.

Proof. Let $A := X(\overline{\mathbb{B}}^2)$ and notice that A is a compact subset of $\mathbb{R}^3$ since $X \in C^0(\overline{\mathbb{B}}^2, \mathbb{R}^3)$. Assume (i) and take a point $x \in B_\theta(A)$. If $x \in A$ simply define $\Pi_A(x) := x$, otherwise one has $\operatorname{dist}(x, A) > 0$, since A is closed. Hence there is at least one point $a \in A$, such that $|x - a| = \operatorname{dist}(x, A)$ by compactness of A. On the other hand, by Theorem 5.2, A is a $C^{1,1}$-submanifold of $\mathbb{R}^3$. In particular, there exists a tangent plane $T_a A$ in a and a function $g \in C^{1,1}(\mathbb{B}^2_{\theta/300}(0))$ with $a = g(z)$ for some $z \in \mathbb{B}^2_{\theta/300}(0)$.

Hence z furnishes a local minimum of the C^1-function $h(\bar{z}) := |x - g(\bar{z})|^2$, $\bar{z} \in B^2_{\theta/300}(0)$. Therefore we have

$$0 = Dh(z) = 2(x - a) \cdot Dg(z),$$

i.e., $x - a \perp T_a A$. Since $\operatorname{dist}(x, A) < \theta$ we conclude that x is contained in the open segment (y_1, y_2), where y_1 and y_2 are the centers of the two touching balls at

a tangent to the affine tangent plane $a + T_a A$. If there were another point $b \in A \setminus \{a\}$, such that $\operatorname{dist}(x, A) = |x - b| < \theta$, then we would infer $b \in B_\theta(y_1) \cup B_\theta(y_2)$ contradicting Corollary 4.2. Thus we have proved reach $A \geq \theta$. The second part of the statement, i.e., reach $X(\partial\mathbb{B}^2) \geq \theta$ follows from $\triangle[X|_{\partial\mathbb{B}^2}] \geq \theta$ alone by [15, Lemma 3 (iii)].

To prove the converse notice first that reach $(X(\partial\mathbb{B}^2)) \geq \theta$ implies (again by [15, Lemma 3 (iii)]) that $\triangle[X|_{\partial\mathbb{B}^2}] \geq \theta$. So for an indirect reasoning we can assume that $\triangle[X] < \theta$, which implies that there are parameters $w \in \mathbb{B}^2$, $w' \in G$ such that

$$0 < \triangle[X] \leq r := r(X(w); X(w'), DX(w')) < \theta,$$

in particular, $X(w) \neq X(w')$ and $X(w) \notin T_{w'}X$ by Definition 2.1. Then $d := \operatorname{dist}(X(w), T_{w'}X)$ satisfies

$$\frac{d}{|X(w) - X(w')|} = \sin\alpha,$$

where

$$\alpha := \sphericalangle(X(w) - X(w'), \pi_{w'}(X(w)) - X(w')) \in [0, \pi/2]$$

is the angle between the vector $X(w) - X(w')$ and the tangent plane $T_{w'}X - X(w')$. On the other hand, by elementary geometry,

$$\frac{|X(w) - X(w')|}{2r} = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha,$$

so that

$$\frac{d}{|X(w) - X(w')|} = \frac{|X(w) - X(w')|}{2r},$$

or

$$\frac{2\operatorname{dist}(X(w), T_{w'}X)}{|X(w) - X(w')|^2} = \frac{1}{r}$$

contradicting (6.1) of Theorem 6.1. □

The final result that we will need in order to investigate sequences $(X_j)_j$ of admissible surfaces $X_j \in \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$, $j \in \mathbb{N}$, having a uniform positive lower bound on their global radius of curvature is Federer's compactness theorem for sets of positive reach:

Theorem 6.3 [10, Thm. 4.13 & Rmk. 4.14]
Let $t > 0$, $K \subset \mathbb{R}^N$ be compact. Then the set

$$\{A \subset K, A \neq \emptyset, \operatorname{reach}(A) \geq t\}$$

is compact with respect to the Hausdorff distance.

6.2 Geometric convergence and compactness

In this section we prove two theorems on compactness and convergence of surfaces with uniform global curvature and area/diameter bounds. We state these results for disk type surfaces only. For closed compact surfaces the situation is in fact simpler, see Theorem 7.2 and the remarks in the next section.

It is known in differential geometry that appropriate uniform bounds on volume, injectivity radius and (some sort of) curvature lead — in *much* greater generality than considered here — to compactness of families of Riemannian metrics, see e.g. Berger's book [4, Sects. 12.4 and 11.5] and references therein, or the recent work of Anderson et al. [1, Sect. 3]. We use similar ideas in a simple context. However, since we work below the C^2-category, any direct use of (classically understood) curvature is impossible. Federer's compactness theorem and the results of Sect. 5 provide replacements for this gap.

Theorem 6.4 *Let $X_j\colon \mathbb{B}^2 \to \mathbb{R}^3$ be a sequence of admissible surfaces with $\triangle[X_j] \geq \theta > 0$. Assume moreover that:*

- (i) $\sup_j \mathcal{H}^2(X_j(\overline{\mathbb{B}}^2)) \leq M < +\infty$;
- (ii) $X_j\big|_{\partial\mathbb{B}^2}$ *are weakly monotone and parametrize rectifiable Jordan curves with global radius of curvature uniformly bounded below by θ, and there exists some $R > 0$ such that each of the curves $X_j(\partial\mathbb{B}^2)$ contains a point $p_j \in B_R(0)$.*

Then one can select a subsequence j' such that $A_{j'} := X_{j'}(\overline{\mathbb{B}}^2)$ converge in Hausdorff distance to A, A is a $C^{1,1}$-manifold with boundary, and the nearest point projection $\pi_A\colon B_\theta(A) \to A$ is well defined.

Corollary 6.5 *Under the assumptions of Theorem 6.4, one can choose a subsequence which satisfies also*

$$\mathcal{H}^2(X_{j'}(\mathbb{B}^2)) \to \mathcal{H}^2(A).$$

Theorem 6.6 *Let a sequence $(X_j)_j \subset \mathcal{A}(\mathbb{B}^2, \mathbb{R}^3)$ satisfy $\triangle[X_j] \geq \theta > 0$ and $\triangle[X_j\big|_{\partial\mathbb{B}^2}] \geq \theta$, and let $X_j\big|_{\partial\mathbb{B}^2}$ be weakly monotone parametrizations of the boundary curves $X_j(\partial\mathbb{B}^2)$. Assume also that $X \in C^0(\overline{\mathbb{B}}^2, \mathbb{R}^3)$ and that $X_j(\partial\mathbb{B}^2)$ converges to $X(\partial\mathbb{B}^2)$ in Hausdorff distance.*

If $X_j(w) \to X(w)$ as $j \to \infty$ for all w belonging to some dense subset of $\mathbb{B}^2$, then

- (i) *the X_j are uniformly bounded;*
- (ii) *the sets $A_j := X_j(\overline{\mathbb{B}}^2)$ converge in Hausdorff distance to a $C^{1,1}$-manifold A with* reach $A \geq \theta$, *and we have $A = X(\overline{\mathbb{B}}^2)$.*

Remarks. 1. It follows from our proofs that in both theorems above the limit manifold A is also equipped with local graph representations whose norms and sizes are uniformly controlled by θ, as described in Sect. 5. Moreover, Corollary 3.4 holds for A, that is, we have two touching balls B_1 and B_2 at every point of A, and $A \subset \mathbb{R}^3 \setminus (B_1 \cup B_2)$.

2. The uniform area bound in the assumptions of Theorem 6.4 is satisfied e.g. when $\sup_j \|X_j\|_{W^{1,2}} \le M < +\infty$. The second part of assumption (ii), i.e. the existence of a point $p_j \in X_j(\partial\mathbb{B}^2) \cap B_R(0)$, is obviously satisfied when the boundary contours converge to a fixed curve or are themselves fixed (as often encountered in the calculus of variations).
3. In fact, it follows from the initial steps of our proof that for any family of admissible surfaces $\mathcal{G}$ such that all $X \in \mathcal{G}$ are weakly monotone on $\partial\mathbb{B}^2$ and satisfy $\min(\triangle[X], \triangle[X\big|_{\partial\mathbb{B}^2}]) \ge \theta > 0$, we have

$$\sup_{X\in\mathcal{G}} \operatorname{diam} X(\mathbb{B}^2) < +\infty \;\Leftrightarrow\; \sup_{X\in\mathcal{G}} \mathcal{H}^2(X(\mathbb{B}^2)) < +\infty. \tag{6.2}$$

Similarly, for any family $\mathcal{C}$ of Jordan curves such that $\triangle[\gamma] \ge \theta > 0$ for all $\gamma \in \mathcal{C}$ we have

$$\sup_{\gamma\in\mathcal{C}} \operatorname{diam} \gamma < +\infty \;\Leftrightarrow\; \sup_{\gamma\in\mathcal{C}} \mathcal{H}^1(\gamma) < +\infty. \tag{6.3}$$

Proof of Theorem 6.4. The overall strategy is as follows. We first show that all $A_j := X_j(\overline{\mathbb{B}}^2)$ are uniformly bounded, and all boundary curves $\gamma_j := X_j(\partial\mathbb{B}^2)$ have uniformly bounded length. Then, applying Federer's compactness theorem, we select subsequences of A_j, γ_j that converge in Hausdorff distance to A and γ. Finally, we cover A and γ by two finite families $\mathcal{F}_A$ and $\mathcal{F}_\gamma$ of open balls, and exploit Theorems 5.1 and 5.2 to see that in each of these balls A_j coincides with a graph (or a piece of graph) of a $C^{1,1}$-function. Hausdorff convergence combined with norm estimates for these functions allows to apply the Arzela–Ascoli compactness theorem and to conclude finally that A is a $C^{1,1}$-manifold with boundary.

We now present the details.

Step 1: all $A_j := X_j(\overline{\mathbb{B}}^2)$ are contained in a fixed ball $B_{\tilde{R}}(0) \subset \mathbb{R}^3$. To see this, fix j and consider the covering $\{B_{\sigma_0\theta}(p) \mid p \in A_j\}$ of A_j, where σ_0 is given by Corollary 5.3. Apply Vitali's lemma to this covering to obtain a (possibly finite) sequence of pairwise disjoint balls $B_{\sigma_0\theta}(p_k)$, where $p_k \in A_j$, such that $\{B_{5\sigma_0\theta}(p_k) \mid k = 1, 2, \ldots\}$ is a covering of A_j. Take N of these balls. Invoking (5.7) for each of them, and summing w.r.t. k, we obtain

$$N K^{-1}\sigma_0^2\theta^2 \le \sum_{k=1}^{N} \mathcal{H}^2(B_{\sigma_0\theta}(p_k) \cap A_j) \le \mathcal{H}^2(X_j(\overline{\mathbb{B}}^2)) \le M.$$

This yields $N \le KM\sigma_0^{-2}\theta^{-2}$, and next $\operatorname{diam}(X_j(\overline{\mathbb{B}}^2)) \le 10KM\sigma_0^{-1}\theta^{-1}$. By (ii), there is some $\tilde{R} > 0$ such that $\bigcup_j X_j(\overline{\mathbb{B}}^2) \subset B_{\tilde{R}}(0)$.

Step 2: length bounds for γ_j. To show that all $\gamma_j = X(\partial\mathbb{B}^2)$ have uniformly bounded length, notice first that

$$\mathcal{H}^1(\gamma \cap B_{\theta/8}(p)) \le \theta \tag{6.4}$$

whenever $p \in \gamma$ and γ is a Jordan curve in $\mathbb{R}^3$ with $\triangle[\gamma] \ge \theta > 0$. Indeed, let Γ be the arc length parametrization of γ. Suppose w.l.o.g. that $\Gamma(0) = 0 = p$,

$\Gamma'(0) = (1,0,0)$. We have the uniform estimate $|\Gamma'(t) - \Gamma'(s)| \le \theta^{-1}|t-s|$, and this implies for the first coordinate Γ_1 of Γ:

$$\Gamma_1'(s) > \frac{1}{2} \qquad \text{for all } s \in (-\theta/2, \theta/2).$$

Thus, estimating the first coordinate of Γ, we obtain

$$\Gamma(\pm\theta/2) \notin B_{\theta/4}(p) \tag{6.5}$$

and therefore

$$\gamma \cap B_{\theta/4}(p) \subset \Gamma([-\theta/2, \theta/2]),$$

which leads to (6.4), since $\Gamma(\tau) \notin B_{\theta/8}(p)$ for all $\tau \notin [-\theta/2, \theta/2]$. To see the latter suppose there existed another curve point $q := \Gamma(\tau) \in B_{\theta/8}(p)$ with $\tau \notin [-\theta/2, \theta/2]$. Then (since Γ is injective due to $\triangle[\gamma] \ge \theta > 0$ and [15, Lemma 1])

$$0 < \operatorname{dist}(q, \Gamma([-\theta/2, \theta/2])) < \theta/8,$$

but $|q - \Gamma(\pm\theta/2)| \ge \theta/8$ by (6.5). This implies that there is a next point $\Gamma(\tau^*)$ with $\tau^* \in (-\theta/2, \theta/2)$ such that

$$\operatorname{dist}(q, \Gamma([-\theta/2, \theta/2])) = |q - \Gamma(\tau^*)|,$$

from which we infer that $q - \Gamma(\tau^*) \perp \Gamma'(\tau^*)$. (Note that Γ is of class $C^{1,1}$ by [26, Thm. 1(iii)].) But this means that $q = \Gamma(\tau)$ is contained in the union of balls of radius θ touching γ in $\Gamma(\tau^*)$ contradicting [26, Thm. 1(iv)(a)].

A uniform length bound is now obtained by an application of Vitali's lemma to the covering of γ_j by balls $B_{\theta/40}(p)$, $p \in \gamma_j$: we select an at most countable subfamily of pairwise disjoint balls $B_{\theta/40}(p_k)$, $p_k \in \gamma_j$, such that larger balls $B_{\theta/8}(p)$, $p_k \in \gamma_j$, cover γ_j. Now, the sum of volumes of $B_{\theta/40}(p_k)$ does not exceed the volume of $B_{R+\theta}(0)$, since each $\gamma_j \subset B_R(0)$ by Step 1. This yields the uniform bound $1 \le k \le K$, where $K \le \text{const}(R+\theta)^3\theta^{-3}$. Next, (6.4) for $\gamma := \gamma_j$ implies that

$$\mathcal{H}^1(\gamma_j) \le \sum_k \mathcal{H}^1(\gamma_j \cap B_{\theta/8}(p_k)) \le K\theta \qquad \text{for all } j.$$

Step 3: Hausdorff convergence of γ_j and A_j. Applying Federer's compactness theorem (Theorem 6.3) twice, which is possible due to Lemma 6.2, we select a subsequence (still labeled by the index j) such that

$$\gamma_j \to \gamma, \quad A_j \to A \qquad \text{in Hausdorff distance,} \tag{6.6}$$

and the reaches of γ and A are $\ge \theta$. Moreover, the uniform length bound for γ_j allows us to use the results of [26, Sect. 4] to conclude that γ is a Jordan curve with $\Delta[\gamma] \ge \theta$ and $C^{1,1}$ arc length parametrization Γ.

Step 4: covering γ and A by good patches. We fix an $\varepsilon > 0$ small, to be specified later on. From now on we assume that

$$\operatorname{dist}_H(\gamma_j, \gamma) + \operatorname{dist}_H(A_j, A) \le \frac{1}{2}\varepsilon\theta \qquad \text{for all } j, \tag{6.7}$$

passing to a subsequence if necessary.

We also suppose w.l.o.g. that $R \geq \theta$, where $A_j \subset B_R(0)$ for all j. To exploit the results of Sect. 5, we construct two finite families of disjoint balls, $\mathcal{F}_\gamma$ and $\mathcal{F}_A$, with the following properties:

(i) $\mathcal{F}_\gamma = \{B_{100\varepsilon\theta}(p_k) \mid p_k \in \gamma \text{ for } k = 1, \dots, K\}$
(ii) $\operatorname{card} \mathcal{F}_\gamma = K \lesssim R^3 (\varepsilon\theta)^{-3}$
(iii) The union $\bigcup_{k=1}^{K} B_{500\varepsilon\theta}(p_k)$ contains the set

$$B_{100\varepsilon\theta}(\gamma) \cup \bigcup_{j=1}^{\infty} B_{99\varepsilon\theta}(\gamma_j), \tag{6.8}$$

i.e., enlarging the balls from $\mathcal{F}_\gamma$ we obtain a covering of tubular neighbourhoods of γ and all γ_j.
(iv) $\mathcal{F}_A = \{B_{4\varepsilon\theta}(q_m) \mid q_m \in A \setminus B_{90\varepsilon\theta}(\gamma) \text{ for } m = 1, \dots, N\}$
(v) $\operatorname{card} \mathcal{F}_A = N \lesssim R^3 (\varepsilon\theta)^{-3}$
(vi) The union $\bigcup_{m=1}^{M} B_{20\varepsilon\theta}(q_m)$ contains the set

$$B_{4\varepsilon\theta}(A \setminus B_{100\varepsilon\theta}(\gamma)). \tag{6.9}$$

and each of the sets

$$B_{\varepsilon\theta}(A_j \setminus B_{99\varepsilon\theta}(\gamma_j)) \qquad j = 1, 2, \dots \tag{6.10}$$

To obtain these families, we apply the Vitali lemma twice, first to the covering $\{B_{100\varepsilon\theta}(p) \mid p \in \gamma\}$ of a tubular neighbourhood of γ, and then to the covering of a tubular neighbourhood of $A \setminus B_{100\varepsilon\theta}(\gamma)$. Conditions (i)–(iii) are immediate; the cardinality bound (ii) is obtained precisely as in Steps 1 and 2 of the proof.

To justify (v), we argue as follows: each q_m is a limit of $q_{jm} \in A_j$, and by (6.7) we can assume that $|q_{jm} - q_m| < \varepsilon\theta/2$ for each m; thus, for each fixed j the balls $B_{3\varepsilon\theta}(q_{jm})$ are pairwise disjoint. Now, an argument analogous to the one carried out in Step 1 above shows that for a fixed small ε the index m can take only finitely many values.

Finally, condition (vi) follows from Vitali's lemma and the inclusions

$$B_{\varepsilon\theta}(A_j \setminus B_{99\varepsilon\theta}(\gamma_j)) \subset B_{2\varepsilon\theta}(A \setminus B_{98\varepsilon\theta}(\gamma)) \subset B_{4\varepsilon\theta}(A \setminus B_{100\varepsilon\theta}(\gamma)),$$

which can be obtained from (6.7) and the triangle inequality. Note that the union of all the balls $B_{500\varepsilon\theta}(p_k)$ and $B_{20\varepsilon\theta}(q_m)$ covers the whole of A.

Step 5: convergence of normals at the centers of small patches. We next select for each j points $p_{jk} \in \gamma_j$ and $q_{jm} \in A_j \setminus B_{99\varepsilon\theta}(\gamma_j)$ so that

$$p_{jk} \to p_k, \quad q_{jm} \to q_m \qquad \text{for all } k, m; \tag{6.11}$$
$$|p_{jk} - p_k| + |q_{jm} - q_m| < \varepsilon\theta/2 \qquad \text{for all } j, k, m. \tag{6.12}$$

Moreover, for each fixed j the union of the balls $B_{501\varepsilon\theta}(p_{jk})$, $1 \leq k \leq K$, covers the tubular neighbourhoods (6.8). Similarly, the union of the balls $B_{21\varepsilon\theta}(q_{jm})$

covers $A_j \setminus B_{99\varepsilon\theta}(\gamma_j)$. This follows from (iii) and (vi) in Step 4, and from (6.12). We also have

$$B_{501\varepsilon\theta}(p_{jk}) \subset B_{502\varepsilon\theta}(p_k), \quad B_{21\varepsilon\theta}(q_{jk}) \subset B_{22\varepsilon\theta}(q_m),$$

since the centers p_{jk} and q_{jm} are close to p_k and q_m, respectively.

Let $n_{jm} \in \mathbb{S}^2$ be normal to $A_j = X_j(\mathbb{B}^2)$ at $q_{jm} =: X_j(w_{jm})$, and let $n'_{jk} \in \mathbb{S}^2$ be normal to A_j at $p_{jk} =: X_j(w'_{jk})$, $w'_{jk} \in \partial\mathbb{B}^2$. Selecting finitely many subsequences, we may assume that for all k, m

$$n_{jm} \to \nu_m \in \mathbb{S}^2 \quad \text{and} \quad n'_{jk} \to \nu'_k \in \mathbb{S}^2 \qquad \text{as } j \to \infty, \tag{6.13}$$

$$|n_{jm} - \nu_m| + |n'_{jk} - \nu'_k| < \delta \qquad \text{for all } j = 1, 2 \ldots \text{ and all } k, m. \tag{6.14}$$

Now, select $\varepsilon = 10^{-6}$; Theorems 5.1 and 5.2 can then be applied in balls of radius smaller than $1000\varepsilon\theta$.

Last step: the structure of A. Fixing $\delta > 0$ sufficiently small in (6.14), and denoting by V_m the cylinder with center q_m, axis parallel to ν_m, diameter $r := 21\varepsilon\theta$ and height $2r$, we invoke Theorem 5.1 to conclude that for each j and m

$$A_j \cap V_m$$

is a nice open patch of the graph of

$$g_{jm} \colon B^2_{21\varepsilon\theta}(q_m) \to \mathbb{R}$$

where $B^2_{21\varepsilon\theta}(q_m)$ denotes a fixed 2-dimensional disk, centered at q_m and contained in the plane passing through q_m and perpendicular to ν_m (i.e. in the plane which is the limit of $T_{w_{jm}}X_j$ as $j \to \infty$). By Theorem 5.1, we have $g_{jm} \in C^{1,1}$, $\|g_{jm}\|_{C^{1,1}} \le \text{const}/\theta$ and $\operatorname{Lip}(g_{jm}) \le 2$. (Notice that for each fixed m we slightly tilt these graphs, to fix their common planar domain.)

Since the Lipschitz norms of all Dg_{jm} are uniformly bounded, by the Arzela–Ascoli theorem we may assume, selecting finitely many subsequences if necessary, that $g_{jm} \to g_m$ as $j \to \infty$ in the C^1-topology, for each fixed m. Moreover, each g_m is of class $C^{1,1}$ since the Lipschitz condition for Dg_{jm} is preserved in the limit $j \to \infty$.

Thus, $A \setminus B_{100\varepsilon\theta}(\gamma)$ is covered by finitely many graphs of $C^{1,1}$-functions g_m. Moreover, it follows form Hausdorff convergence of $A_j \to A$ and $\gamma_j \to \gamma$, and from the C^1-convergence $g_{jm} \to g_m$ that each point in this part of A has a neighbourhood U (whose size scales like θ) such that $A \cap U$ is in fact a graph of a $C^{1,1}$-function from a disk to $\mathbb{R}$.

We repeat a similar argument at the boundary, applying Theorem 5.2 instead of Theorem 5.1. As a result, we obtain $C^{1,1}$-functions

$$f_{jk} \colon B^2_{501\varepsilon\theta}(p_k) \to \mathbb{R},$$

defined on disks in the planes that pass through p_k and are perpendicular to ν'_k, such that pieces of A_j in small neighbourhoods of p_k are equal to graphs of f_{jk} restricted to Ω^+_{jk} given by Theorem 5.2. Passing to subsequences as before, we conclude that for each k the set $A \cap B_{501\varepsilon\theta}(p_k)$ is $C^{1,1}$-diffeomorphic to a semidisk.

The whole proof is complete now. □

Proof of Corollary 6.5. Since A is locally a uniform C^1-limit of graphs, we can express the area of A and $X_j(\partial\mathbb{B}^2)$ for $j \gg 1$ using a fixed partition of unity in a fixed open covering of A and the standard formula for the area of a graph of a C^1-function. This obviously yields convergence $\mathcal{H}^2(X_j(\mathbb{B}^2)) \to \mathcal{H}^2(A)$ along the subsequence that was selected in the previous proof. □

Proof of Theorem 6.6.
(i) Choose $R > 2\theta > 0$ so that

$$X(\overline{\mathbb{B}}^2) \subset B_R(0) \subset \mathbb{R}^3. \tag{6.15}$$

If the X_j were not uniformly bounded we could find $p_j \in A_j := X_j(\overline{\mathbb{B}}^2)$ with $|p_j| \geq 2R$ for infinitely many $j \in \mathbb{N}$. In fact, by (6.15) and the Hausdorff convergence $X_j(\partial\mathbb{B}^2) \to X(\partial\mathbb{B}^2)$ we know that p_j is an interior point, i.e., $p_j \in X_j(\mathbb{B}^2)$ for all but finitely many j. For this subsequence we choose arbitrary boundary points $q_j \in X_j(\partial\mathbb{B}^2)$. Since $X_j \in C^0(\overline{\mathbb{B}}^2, \mathbb{R}^3)$ for all j we find continuous curves $\gamma_j \in C^0([0,1], A_j)$ connecting p_j and q_j, i.e., with $\gamma_j(0) = p_j$ and $\gamma_j(1) = q_j$ for each $j \in \mathbb{N}$. By Lemma 5.4 all A_j are orientable and we can find a continuous normal field ν_j on A_j for each j. Consider for each j the two arcs

$$c_j^1 := (\gamma_j + \theta\nu_j \circ \gamma_j)([0,1]),$$
$$c_j^2 := (\gamma_j - \theta\nu_j \circ \gamma_j)([0,1]),$$

which satisfy

$$\operatorname{dist}\left(c_j^i, A_j\right) = \theta \quad \text{for} \quad i = 1, 2, \quad \text{for all} \quad j \in \mathbb{N} \tag{6.16}$$

according to (4.3) of Corollary 4.2 applied to X_j.

Now we claim that there is a number $r > 0$ independent of j such that c_j^1 and c_j^2 can be connected to form a simple closed curve c_j such that

$$B_r(c_j) \cap A_j \subset B_r(p_j) \subset \mathbb{R}^3 \setminus B_{2R-r}(0) \quad \text{for all} \quad j \in \mathbb{N}, \tag{6.17}$$

$$B_r(c_j) \cap X(\partial\mathbb{B}^2) = \emptyset \quad \text{for all} \quad j \gg 1. \tag{6.18}$$

Indeed, connect the points $c_j^1(0) = p_j + \theta\nu_j(p_j)$ with $c_j^2(0) = p_j - \theta\nu_j(p_j)$ by a straight segment σ_j (see Fig. 2) which according to Corollary 4.2 (applied to X_j) satisfies

$$\sigma_j \cap A_j = \{p_j\} \quad \text{for all} \quad j \in \mathbb{N}. \tag{6.19}$$

To join the other two endpoints, i.e., $c_j^1(1) = q_j + \theta\nu_j(q_j)$ with $c_j^2(1) = q_j - \theta\nu_j(q_j)$ without intersecting A_j we construct a curve τ_j in the normal plane[1] P_j to $X_j(\partial\mathbb{B}^2)$ in q_j as follows: Consider the squares $C_{4r}^j \subset P_j$ centered at q_j with two edges parallel and two edges perpendicular to ν_j and edge length $8r \leq \theta/300$ so small that

$$\operatorname{dist}\left(C_{4r}^j, \psi_j(\mathbb{R}) \cap B^2_{\theta/300}(0)\right) = 4r. \tag{6.20}$$

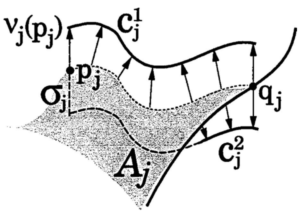

Fig. 2 The arcs c_j^1, c_j^2 and the segment σ_j passing through p_j

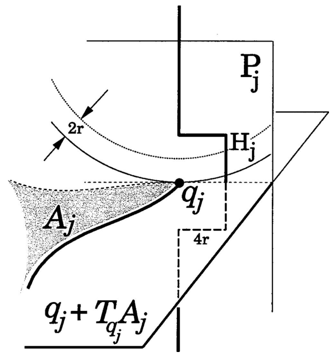

Fig. 3 The curve c_j in the neighbourhood of q_j. Replacing a piece of the normal line by H_j, we link c_j with the boundary curve of A_j. Uniform global curvature bounds yield uniform estimates for the distance of both curves

Here $B^2_{\theta/300}(0) \subset q_j + T_{q_j}A_j$, and $\psi_j \in C^{1,1}(\mathbb{R})$ is the function determining the boundary of the local graph-representation of A_j obtained by applying Theorem 5.2 to X_j in $q_j \in X_j(\partial\mathbb{B}^2)$. The uniform estimate $\|\psi_j\|_{C^{1,1}} \le C/\theta$ allows us to choose r independent of j. The normal line through $\nu_j(q_j)$ cuts C^j_{4r} into two halfs H^j, $\bar{H}^j$ one of which, say H^j, intersecting $B^2_{\theta/300}(0) \subset q_j + T_{q_j}A_j$ in $\Omega_j^- := \{(x, y) \in B^2_{\theta/300}(0) : y < \psi_j(x)\}$. This implies by Theorem 5.2 and (6.20) that

$$\operatorname{dist}(H^j, A_j) \ge 2r. \tag{6.21}$$

Otherwise we could find points $a \in A_j$ and $c \in H^j$ with

$$|c - a| < 2r, \tag{6.22}$$

which implies by (4.3) of Corollary 4.2 and elementary geometry (see Fig. 3) that

$$c \notin \overline{B_{\theta-2r}(q_j \pm \theta\nu_j(q_j))}.$$

[1] Note that since $\triangle[X_j|_{\partial\mathbb{B}^2}] \ge \theta > 0$, the tangent direction and the normal plane exist everywhere along the curve $X_j(\partial\mathbb{B}^2)$, see [15, Lemma 2].

Hence c lies on the segment of H^j parallel to ν_j, which implies

$$\pi_j(c) = H^j \cap (q_j + T_{q_j} A_j),$$

where π_j denotes the orthogonal projection onto the affine tangent plane $q_j + T_{q_j} A_j$. Now (6.21) leads to

$$\operatorname{dist}(\pi_j(c), B^2_{\theta/300} \setminus \Omega_j^-) \geq 2r,$$

but (6.22) on the other hand would give $|\pi_j(c) - \pi_j(a)| < 2r$, a contradiction, since $\pi_j(a) \in B^2_{\theta/300} \setminus \Omega_j^-$. Hence (6.21) holds true.

Now connect $c_j^1(1)$ and $c_j^2(1)$ with the endpoints $q_j \pm 4r\nu_j(q_j)$ of H^j by straight segments of length $\theta - 4r$ to obtain τ_j. By (4.3) of Corollary 4.2 applied to X_j and by (6.21) we find

$$\operatorname{dist}(\tau_j, A_j) \geq 2r \quad \text{for all} \quad j \in \mathbb{N}, \tag{6.23}$$

which, together with (6.16) and (6.19) implies (6.17). Relation (6.18) follows from (6.15), (6.23), (6.16) and (6.19) as well by virtue of the Hausdorff convergence $X_j(\partial\mathbb{B}^2) \to X(\partial\mathbb{B}^2)$ as $j \to \infty$, since $8r < \theta/300 < R/600$.

This convergence and (6.21) imply also that the curves c_j are non-trivially linked with $X(\partial\mathbb{B}^2)$ for j sufficiently large, because c_j is linked with $X_j(\partial\mathbb{B}^2)$ for each $j \in \mathbb{N}$ by construction. Therefore $c_j \cap X(\mathbb{B}^2) \neq \emptyset$ for all $j \gg 1$ as $X \in C^0(\overline{\mathbb{B}}^2, \mathbb{R}^3)$ is of disk-type.

Let $X(w_j) \in c_j \cap X(\mathbb{B}^2)$. We may take a subsequence to obtain $w_j \to w \in \overline{\mathbb{B}}^2$. If $w \in \partial\mathbb{B}^2$, then $X(w) \in X(\partial\mathbb{B}^2)$, and by continuity

$$\operatorname{dist}(X(w_j), X(\partial\mathbb{B}^2)) < r \qquad \text{for all } j \gg 1,$$

contradicting (6.18). Hence $w \in \mathbb{B}^2$. Then $B_{r/4}(X(w)) \subset B_{r/2}(c_j)$ for all $j \gg 1$. By continuity of X we find $\delta > 0$ such that $X(B_\delta(w)) \subset B_{r/2}(c_j)$ for all $j \gg 1$. Since we have assumed pointwise convergence $X_j \to X$ on a dense subset of $\mathbb{B}^2$ we find $w' \in B_\delta(w)$ such that $X_j(w') \to X(w')$ as $j \to \infty$, in particular

$$X_j(w') \in B_r(c_j) \quad \text{for all} \quad j \gg 1. \tag{6.24}$$

On the other hand by (6.15)

$$X_j(w') \in B_{R+r}(0) \quad \text{for all} \quad j \gg 1. \tag{6.25}$$

But (6.24) together with (6.25) contradict (6.17) as $r < R/4800$, which concludes the proof of (i).

(ii) Since the X_j are uniformly bounded by (i) we may apply Federer's compactness theorem, Theorem 6.3, to get a subsequence $A_j := X_j(\overline{\mathbb{B}}^2)$ converging in Hausdorff distance to a set $A \subset \mathbb{R}^3$ with reach $A \geq \theta$, which implies that A is closed (cf. [10, Rmk. 4.2]). Since $X_j \to X$ on a dense subset of $\mathbb{B}^2$ as $j \to \infty$ we obtain $X(w) \in A$ for all w contained in that dense subset of $\mathbb{B}^2$. By continuity we find $X(\overline{\mathbb{B}}^2) \subset A$, since A is closed.

We claim that $X(\overline{\mathbb{B}}^2) = A$. If there were a point $p \in A \setminus X(\overline{\mathbb{B}}^2)$, then we could find $r > 0$ such that

$$B_r(p) \cap X(\overline{\mathbb{B}}^2) = \emptyset, \tag{6.26}$$

because $X(\overline{\mathbb{B}}^2)$ is closed. There are points $p_j = X_j(w_j) \in A_j := X_j(\overline{\mathbb{B}}^2)$ such that $p_j \to p$ as $j \to \infty$, and according to Corollary 4.2 also the normal lines $l_j = l_j(w_j)$ converge to a limit direction $l = l(w)$ if we take an appropriate subsequence with $w_j \to w \in \overline{\mathbb{B}}^2$.

We claim that

$$A \cap B_1 = A \cap B_2 = \emptyset \tag{6.27}$$

for the two open balls B_1, B_2 centered on l with radius θ touching each other at p. Indeed, if (6.27) were not true, say if $q \in A \cap B_1$ we could find $q_j \in A_j$ close to q, such that $q_j \in A_j \cap B_1^j$, where B_1^j is one of the two excluded balls B_1^j, B_2^j centered on l_j with radius θ touching each other in p_j for j sufficiently large, thus contradicting Corollary 4.2. (Here we used the Hausdorff convergence $B_1^j \to B_1$, $B_2^j \to B_2$ as $p_j \to p$, $l_j \to l$ for $j \to \infty$, and the Hausdorff convergence $A_j \to A$ to find $q_j \to q$.)

According to (6.26) and (6.27) one obtains segments $I_r := \{p + t\nu(w) : |t| < r/2\}$, where $\nu(w) \in \mathbb{S}^2$ is parallel to $l = l(w)$, with

$$\operatorname{dist}(I_r, X(\overline{\mathbb{B}}^2)) \geq r/2.$$

Consequently, for $I_r^j := \{p_j + t\nu_j(w) : |t| < r/2\}$, where $\nu_j(w) \in \mathbb{S}^2$ is parallel to $l_j = l_j(w_j)$, we have

$$\operatorname{dist}\left(I_r^j, X(\overline{\mathbb{B}}^2)\right) \geq r/4 \quad \text{for all} \quad j \gg 1,$$

since $I_r^j \to I_r$ in Hausdorff distance as $j \to \infty$.

Similarly as in the proof of Part (i) we can find $\rho < r/8$ independent of j and closed curves c_j containing the segments I_r^j such that

$$B_\rho(c_j) \cap A_j \subset B_\rho(p_j) \subset B_{r/8}(p_j) \quad \text{for all} \quad j \in \mathbb{N}, \tag{6.28}$$

$$B_\rho(c_j) \cap X(\partial\mathbb{B}^2) = \emptyset \quad \text{for all} \quad j \gg 1. \tag{6.29}$$

As in Part (i) we can argue that the curves c_j are non-trivially linked with $X(\partial\mathbb{B}^2)$ for $j \gg 1$, so that

$$c_j \cap X(\mathbb{B}^2) \neq \emptyset \quad \text{for all} \quad j \gg 1.$$

In particular, there are $\bar{w}_j \to \bar{w} \in \overline{\mathbb{B}}^2$ with $X(\bar{w}_j) \in c_j \cap X(\mathbb{B}^2)$ for an appropriate subsequence. As in the proof of (i) the relation (6.29) implies that $\bar{w} \in \mathbb{B}^2$, and $B_{\rho/4}(X(\bar{w})) \subset B_{\rho/2}(c_j)$ for $j \gg 1$. By continuity of X we find $\delta > 0$ such that $X(B_\delta(\bar{w})) \subset B_{\rho/2}(c_j)$ for all $j \gg 1$. Due to the pointwise convergence $X_j \to X$ on a dense subset of $\mathbb{B}^2$ there exists $w' \in B_\delta(\bar{w})$ such that $X_j(w') \to X(w')$ as $j \to \infty$, in particular by (6.28)

$$X_j(w') \in B_\rho(c_j) \subset B_\rho(p_j) \subset B_{r/8}(p_j) \quad \text{for all} \quad j \gg 1.$$

Now take j even larger if necessary to have

$$|X_j(w') - X(w')| < r/4, \quad |p_j - p| < r/4$$

to obtain $X(w') \in B_{5r/8}(p)$ which contradicts (6.26). Thus we have shown that $X(\overline{\mathbb{B}}^2) = A$.

We are now in the situation of Theorem 6.4, where assumption (i) was applied only to derive a uniform diameter bound for the $X_j(\overline{\mathbb{B}}^2)$, which we have shown in Part (i) of the present theorem. Hence $A = X(\overline{\mathbb{B}}^2)$ is a $C^{1,1}$-manifold with boundary. Since X was fixed from the beginning we may conclude from the subsequence principle that the *whole* sequence A_j converges to A in Hausdorff distance. □

7 Ideal surfaces

In this section we apply the previously obtained regularity and compactness results to prove the existence of the so-called ideal surfaces.

Fix $\theta > 0$, $g \geq 1$ and a closed, compact reference surface M_g of genus g that is smoothly embedded in $\mathbb{R}^3$ in such a way that its tubular neighbourhood has width $\geq \theta$ (i.e. the nearest point projection $\Pi_M : \bigcup_{p \in M} B_\theta(p) \to M$ is well defined). We consider the class

$$\mathcal{S}_\theta(M_g) = \{X \in \mathcal{A}(M_g, \mathbb{R}^3) : \triangle[X] \geq \theta, \ X(M_g) \overset{\text{iso}}{\simeq} M_g\} \tag{7.1}$$

of thick admissible surfaces that are (ambiently) isotopic to M_g. (See e.g. [16, Vol. 5, p. 209] for the definition of isotopy.) Clearly, $\mathcal{S}_\theta(M_g)$ is nonempty as the identity mapping belongs to it.

Theorem 7.1 *For each $g = 1, 2, \ldots$, each $\theta > 0$ and each fixed reference surface M_g satisfying the above assumptions, the class $\mathcal{S}_\theta(M_g)$ contains a surface of minimal area.*

Basically, this result follows from the geometric compactness, Theorem 6.4 in the previous section and its Corollary 6.5. However, these two results have been stated only for disk type surfaces, therefore we indicate briefly the changes that need to be introduced in various regularity results in Sects. 3–5, to adapt them to compact closed surfaces of genus $g \geq 1$.

Remark 1. The stretching property, Lemma 3.1. The claim of this lemma still holds at every good parameter $w \in M_g$; we can take $\varrho_0 := \theta$ as there is no boundary. Indeed, $\mathbb{R}^3 \setminus X(M_g)$ has two connected components U_1 and U_2; see e.g. Lima's paper [20] for a clever proof for general compact closed hypersurfaces in $\mathbb{R}^{d+1}$. We have $\partial U_1 = \partial U_2 = X(M_g)$. One of the excluded balls B_1, B_2 at $X(w)$ is contained in U_1, and the other one in U_2. Thus, any segment $I \subset \mathbb{R}^3 \setminus (B_1 \cup B_2)$ with one endpoint on ∂B_1 and the other endpoint on ∂B_2 must contain at least one point of the common boundary of U_1 and U_2, i.e., a point of the surface.

Remark 2. The proofs of interior continuity of the normal (cf. Sect. 3) and the description of the local graph structure (cf. Sect. 5) of $X(M_g)$ remain unchanged. All the arguments there rely only on the stretching property, on the existence of excluded touching balls, and on elementary geometry. In fact, everything simplifies a bit, as there is no boundary. In particular, for each point $p \in X(M_g)$ the intersection of $V_{\theta/100}(p)$ and $X(M_g)$ is isometric to the graph of a $C^{1,1}$-function f with $\mathrm{Lip}\, f \le 1$, $\|f\|_{C^{1,1}} \le C/\theta$. The area estimates of Corollary 5.3 also hold true.

After these preparations, we are ready to state and prove the following.

Theorem 7.2 *Each sequence of surfaces $X_j \in \mathcal{S}_\theta(M_g)$, $j = 1, 2, \ldots$, such that*

$$\sup_{j\in\mathbb{N}} \mathcal{H}^2(X_j(M_g)) < +\infty$$

contains a subsequence $X_{j'}$ such that

(i) *$X_{j'}(M_g)$ converge (in Hausdorff distance and in C^1-topology) to a closed compact $C^{1,1}$-surface $A \subset \mathbb{R}^3$ that satisfies*

$$\mathrm{reach}\, A \ge \theta;$$

(ii) *$\mathcal{H}^2(X_{j'}(M_g)) \to \mathcal{H}^2(A)$ as $j' \to \infty$;*

(iii) *A has genus g and moreover $A \overset{\mathrm{iso}}{\simeq} X_{j'}(M_g)$ for all j' sufficiently large.*

Proof. Conditions (i) and (ii) are obtained precisely in the same way as Theorem 6.4 and Corollary 5.3. In fact, the proof of compactness shortens a bit, as we do not have to care about the boundaries of surfaces. (One just needs Step 1, i.e., a uniform diameter bound for all $A_j := X_j(M_g)$, then an application of Federer's compactness theorem yields convergence of some subsequence of A_j in the Hausdorff distance, next a good finite covering is constructed by a simplified version of Step 3, and finally, passing to convergent subsequences of normals and applying Arzela–Ascoli's theorem, we obtain local C^1-convergence of graph patches that cover A_j to graph patches that cover A.)

Thus, we only have to prove (iii). This is done in two steps. For sake of simplicity we denote the selected subsequence again by X_j.

Step 1. The genus of A is equal to g, as A is homotopically equivalent to $X_j(M_g)$ for all j sufficiently large (see Dubrovin, Novikov and Fomenko, [8, Ch. 17], for the definition and its consequences). To check the equivalence, we fix j so large that

$$\mathrm{dist}_H(A_j, A) < \frac{\theta}{2}. \tag{7.2}$$

Let

$$f := \Pi_A\big|_{A_j}, \quad g := \Pi_{A_j}\big|_A. \tag{7.3}$$

We shall show that $f \circ g \simeq \mathrm{id}_A$ and $g \circ f \simeq \mathrm{id}_{A_j}$, where $\simeq$ denotes the homotopy of maps.

For every $x \in A$ we have

$$|x - g(x)| < \theta/2, \quad |g(x) - g(f(x))| < \theta/2$$

by (7.2). Hence, $|x - f(g(x))| < \theta$ and the whole closed segment with endpoints x, $f(g(x))$ is contained in $B_\theta(A)$, i.e. in the domain of Π_A. The homotopy $f \circ g \simeq \mathrm{id}_A$ is given by

$$H_t(x) = \Pi_A\big(tx + (1-t)f(g(x))\big) \qquad \text{for } x \in A \text{ and } t \in [0,1].$$

The homotopy $g \circ f \simeq \mathrm{id}_{A_j}$ is constructed in a similar way.

Step 2. There exists an isotopy that carries a neighbourhood of A to a neighbourhood of A_j (and A to A_j) for $j \gg 1$.

Set, for $p \in \mathbb{R}^3$ and $\nu \in \mathbb{S}^2$, $I_r(p, \nu) := \{p + t\nu \colon |t| < r/2\}$. Let

$$N_j := B_{\theta/2}(A_j) = \bigcup_{p \in A_j} I_\theta(p, \nu_j(p)), \tag{7.4}$$

where ν_j is a fixed vector field normal to $A_j = X_j(M_g)$, and

$$\widetilde{N}_j := \bigcup_{p \in A_j} I_\theta\Big(\big(\Pi_{A_j}\big|_A\big)^{-1}(p), \nu_j(p)\Big). \tag{7.5}$$

To prove that $\widetilde{N}_j$ is an open neighbourhood of A and to define the desired isotopy, we check first that

$$\Pi_{A_j}\big|_A \colon A \to A_j$$

is bijective for all $j \gg 1$.

To check injectivity, assume the contrary: for infinitely many j there exist points $p_1^j \neq p_2^j \in A$ with $\Pi_{A_j}(p_1^j) = \Pi_{A_j}(p_2^j) = q_j \in A_j$. Thus, by the triangle inequality,

$$|p_1^j - p_2^j| \leq 2\,\mathrm{dist}_H(A_j, A) \to 0 \qquad \text{as } j \to \infty.$$

Passing to a subsequence if necessary, we can assume that $p_1^j, p_2^j \to p \in A$ as $j \to \infty$. Now, p_1^j, p_2^j belong to the line $\ell_j(q_j)$ normal to A_j at q_j. It follows from the C^1-convergence of local graph patches of A_j to graph patches of A that the $\ell_j(q_j)$ is close to $\ell(p_1^j)$, the normal to A at p_1^j. In particular, if $\delta > 0$ is fixed, then the excluded balls associated to A at p_1^j contain two segments of $\ell_j(q_j)$ having common length at least $4\theta - \delta$ for $j \gg 1$. Therefore, p_2^j belongs to one of these balls and is different from p_1^j, a contradiction.

To check surjectivity, consider

$$J_j = \{x \in A_j \mid \exists p(x) \in A : \Pi_{A_j}(p(x)) = x\}.$$

This set is nonempty as $A \subset B_\theta(A_j)$ for all j large. Closedness of J_j follows immediately from closedness of A_j and A and continuity of the projection. Finally,

$$J_j = \{x \in A_j \mid x = \Pi_{A_j}(A \cap I_{2\theta}(x, \nu_j(x))\}$$

and A intersects, for $j \gg 1$, all 'needles' $I_\theta(x, \nu_j(x))$ transversally because of the C^1-convergence proven in (i), so if it intersects one of them, it must intersect all nearby ones. Thus, J_j is open in A_j. As A_j is connected, we have $J_j = A_j$, i.e., $\Pi_{A_j}\big|_A$ is surjective onto A_j.

This implies that $\widetilde{N}_j$ is an open neighbourhood of A for $j \gg 1$.

We can now finally define $\Phi\colon \widetilde{N}_j \times [0,1] \to \mathbb{R}^3$ by

$$\Phi(x,\tau) = x + \tau\left[\Pi_{A_j}(x) - \left(\Pi_{A_j}\big|_A\right)^{-1}(\Pi_{A_j}(x))\right]. \tag{7.6}$$

Thus, the map $\Phi(\cdot,\tau)$ restricted to an arbitrary line orthogonal to A_j agrees with a translation. It is a straightforward exercise to check that

(a) $\Phi(\widetilde{N}_j, 1) = N_j$,
(b) $\Phi(\cdot, 0) = \mathrm{id}_{\widetilde{N}_j}$,
(c) $\Phi(\cdot,\tau)$ is, for each $\tau \in [0,1]$, a homeomorphism whose inverse is given by

$$\xi \mapsto \xi - \tau\left[\Pi_{A_j}(\xi) - \left(\Pi_{A_j}\big|_A\right)^{-1}(\Pi_{A_j}(\xi))\right],$$

(d) $\Phi(A, 1) = A_j$.

Hence, Φ is the desired isotopy. □

Proof of Theorem 7.1. We pick a minimizing sequence and apply Theorem 7.2. (One can parametrize the limiting surface by a composition of X_j and $\Phi(\cdot,1)^{-1}$.) □

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